Special relativity: the union of space and time

Reference webpages:

http://en.wikipedia.org/wiki/Special_relativity
http://en.wikipedia.org/wiki/Thought_experiment
and Chapter 1 in Thorne

Special relativity is a generalization of Galilean relativity that allows comparison of reference frames that move past each other at close to the speed of light. The lion's share of the credit for development goes to Einstein, but many others contributed as well. Starting with relativity, and proceeding even far more so with quantum mechanics, physical theories left the realm of everyday experience into a domain where our common sense does not work at all. This has caused many people to be very uncomfortable with these theories, and it is no surprise that as a result relativity is a favorite target of crackpots. Here we will discuss the principles of special relativity (which is "special" because it only deals with reference frames moving past each other at constant speed) and some of its paradoxes. Questions to keep in mind are:

1. Must science conform to common sense?
2. If, in circumstances beyond our everyday experience, people have different intuitions about the operation of physics, how can this be communicated to people who have different intuition or experience?
3. In a change of perspective such as is represented by relativity or quantum mechanics, how much actually changes in our perspective of the universe? Are our previous deductions still applicable?
4. Fundamentally, how objective is science?

Our third official debate will be held on the third of these topics.

The life of Einstein

Einstein was born in Ulm, Germany in 1879. It is often relayed that he was a poor student in elementary school, but in fact apart from a slow development of speech he was actually excellent; the confusion arises from a shift of how the marks were given, with what were top marks during Einstein's time reversed in later marking systems. At the age of about five, Einstein's father showed him a pocket compass, and Einstein realized that despite the apparent empty space something must cause the compass to move. This began his physics career, in which he retained a remarkably independent spirit of inquiry. His resentment of the regimented teaching of the day meant that although he enjoyed mathematics and physics and did well in them, he failed examinations in other subjects. One result was that after obtaining a diploma from the Polytechnic program at Zurich in 1900 he struggled to find employment, and after two years was only able to become an assistant examiner in the patent office in Bern, Switzerland. Remarkably, however, this sped his development as a scientist because he had to get to the physical heart of patent applications. During this period he published many influential articles, including those in his "miracle year" of 1905 when he published seminal papers on Brownian motion, the photoelectric effect (for which he got the Nobel Prize; he never got it for special or general relativity), and special relativity.

After the patent office experience his fame rose gradually until he was appointed to various academic positions (Zurich, Prague, Zurich again, and Berlin), and in 1933 moved to the Institute for Advanced Study in Princeton, NJ, largely as a reaction to the growing threat of Nazism in Germany. In his later life, he became best known as a pacifist who nonetheless signed a letter to President Roosevelt in 1939 urging the development of the atomic bomb to combat the threat posed by Germany and Japan.

Einstein's scientific contributions are profound and far too varied to describe in a short biography. Suffice it to say that if most physicists were asked to named the greatest physicist of all time, they would name either Einstein or Newton. Even so, he spent most of the last thirty years of his life arguing philosophically against quantum mechanics, which is the most quantitatively successful scientific theory ever developed.

Here we are going to focus on the radical change of perspective initiated by Einstein (mainly) and others related to what is called *special relativity*. We will start with statements about what it means, but will use mathematics to describe it because only in this way can we be precise, and only in this way can we be clearest about its apparent paradoxes.

Philosophy

First, let's start with a little philosophy. After the fact, it is easy to present physical principles as if they are self-evident and derivable from pure mathematics. This is not the case. We can marvel at the brilliance of Einstein and the other pioneers of relativity, and appreciate the philosophical way that they drew their conclusions, but to be scientific one must at some point have contact with experiments. Therefore, ultimately, we have to point to the universe as a whole (or at least, what we've probed observationally) to argue that the theory is correct.

A second philosophical point that many people mistakenly derive from relativity, probably because of the name of the theory, is that the essential point is "everything is relative". In fact, one of the postulates of relativity, and one of its deepest points, is that there are some quantities that are *invariant*, meaning that all observers will measure the same value for those quantities. We'll try to emphasize such invariants when we derive aspects of special relativity.

Galilean Relativity

We should also not get the idea that Einstein was the first one to suggest a principle of relativity. In fact, as we saw earlier, Galileo used thought experiments quite similar to Einstein's to show that something coasting along at a constant velocity should experience all the same local effects as something at rest. He asked his readers to consider experiments performed by someone in a ship's cabin if the ship is moving at a constant speed. He notes that a ball tossed straight up will appear to come straight down; a tank of water will remain level; and in general the experimenter will not be able to tell that the ship is moving. From our standpoint a more familiar and extreme example is traveling in a plane. We might be going 75% of the speed of sound relative to the ground, but we can still be served bad food without it ending up in our faces!

Put more formally, all local experiments we do in an inertial frame will turn out the same independent of our velocity relative to a given frame. However, note the restrictions to *local* experiments and *inertial* frames. If you somehow opened the window of your plane and stuck your head out, it would be the last thing you ever did; there is a quite clear difference in physical effects when you have contact with other frames! In addition, when the plane accelerates (e.g., by hitting turbulence) it is sickeningly clear that you are not at rest. In more benign situations, such as experiments on a rotating Earth, the non-inertial nature of the frame leads one to introduce fictitious forces such as the Coriolis force.

How, then, would we phrase Galilean relativity mathematically? A useful way to do this is to consider two observers moving at a constant velocity v relative to each other. Let us set up Cartesian coordinate systems for both: for one frame the coordinates are (t, x, y, z) and for the other are (t', x', y', z'). We will refer to these as, respectively, the unprimed and primed frames. Here t means time, and we will make our lives easier by ensuring that the x axis is parallel to the x' axis, and similarly for y and z.

Suppose that, as seen in the unprimed system, the primed system is moving in the $+x$ direction with speed v. Note that we can always rotate our coordinate axes so that the x axis lines up with this speed; if you prefer making your algebra messier you can always do it more generally, but we won't bother. If we set up our initial conditions so that at time $t = t' = 0$ we have $x' = 0$ (i.e., the origins of the two systems are coincident), this implies that at time t, the origin of the primed system is at $x = vt$ as measured in the unprimed system. Of course, in the primed system, the origin is always at $x' = 0$. In addition, the perpendicular directions y and z are equal to their primed counterparts, and $t = t'$.

Therefore, the coordinate transformation for Galilean relativity becomes

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \; . \end{aligned} \tag{1}$$

We also find that Newton's laws of motion are invariant in form under these transformations. This is as expected, and is a consequence of our inability to tell whether we are moving steadily or not from purely local experiments. Among other things, this law tells us how velocities should add. Consider, for example, something that moves with speed u in the x direction as seen in the unprimed frame. Therefore, $dx/dt = u$. In the primed frame we have

$$u' = dx'/dt' = d(x - vt)/dt = dx/dt - v = u - v \; . \tag{2}$$

This is the simple, intuitive result. If a train goes by me at 100 km/hr and I throw a baseball parallel to the train at 100 km/hr, someone inside the train sees the ball not moving in that direction at all. If I throw antiparallel to the train at 100 km/hr, the person in the train sees a speed of 200 km/hr. Note, by the way, that if we want to transform from the primed frame to the unprimed frame, all we have to do is reverse the sign of v and switch the primed with unprimed variables. Very simple.

The Problem with Maxwell's Equations

In the mid-1800s, however, a problem emerged. After many people had for several decades experimented with electricity and magnetism, James Clerk Maxwell came up with a compact set of equations that beautifully described all the phenomena. To this day, Maxwell's achievement ranks among the very greatest in the history of physics. Surprisingly, though, Maxwell's equations are *not* invariant under a Galilean transformation. For example, a blatant contradiction emerges when one tries to determine the speed of light in different frames with this theory. According to this theory, the propagation speed was the same whether the source was moving or not, which violates the velocity addition law that we derived above. This is a result that could be obtained if light propagated through a medium, similarly to how the speed of sound is independent of the motion of the source (although the frequency isn't). However, the famous Michelson-Morley experiment found no evidence of any "luminiferous ether" through which light traveled. Given the overwhelming success of Newton's theories in the previous two centuries a number of people very logically tried to find formulations of Maxwell's equations that obeyed Galilean relativity. None, however, could be squared with experiment. What could be done?

The Postulates of Special Relativity

Lorentz, Poincaré, Fitz-Gerald, and others suggested essentially ad hoc ways of explaining the above results. Einstein, however, was the one who put it on a more axiomatic footing, which is why we reasonably give him the lion's share of the credit. He suggested two postulates:

- The laws of physics as derived from local experiments are the same for all inertial observers.
- All such observers measure the same speed for light in a vacuum.

The first postulate is the same one as before. The second, however, seems contradictory; how is it reconciled with normal velocity addition?

To understand this, and to adopt a perspective that has tremendous utility in general relativity, we will consider the fundamental concept of the *invariant interval.* As our first step, recall distance invariance in Euclidean geometry. Suppose we have two points in a three-dimensional space, and in a particular Cartesian coordinate system the points have coordinates (x, y, z) and $(x + dx, y + dy, z + dz)$. For the situations we consider here, dx, dy, and dz need not be infinitesimal quantities, but we write it this way for later compatibility with general relativity (where it is clearest to restrict oneself to infinitesimal distances). The distance ds between the two points is then given by

$$ds^2 = dx^2 + dy^2 + dz^2 \ . \tag{3}$$

This distance is absolutely invariant with respect to coordinate transformations. If you rotate the axes to some new x', y', z' then in general $dx' \neq dx$ and so on, but $ds'^2 = dx'^2+dy'^2+dz'^2 = dx^2 + dy^2 + dz^2 = ds^2$. This is also true if you go for a non-Cartesian system, e.g., spherical polar coordinates. The separation is an *invariant*.

What about when time is involved? Einstein's second postulate says that the distance light travels in a given time is measured to be the same in all frames. It's easier to deal with the squares of distances, so if at time $t = t' = 0$ the ray started out at the origin of the unprimed and primed systems (where, remember, the primed system can move relative to the unprimed system), we would find that

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= c^2dt^2 \\ dx'^2 + dy'^2 + dz'^2 &= c^2dt'^2 \ , \end{aligned} \tag{4}$$

where c is the speed of light in a vacuum.

In fact, let's make a powerful generalization of this. Define an *event* to be something at a specific place and time, which must therefore be designated by four coordinates (t, x, y, z). Consider a nearby event $(t+dt, x+dx, y+dy, z+dz)$, and let the four-dimensional "interval"

ds between the two events be given by

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{5}$$

We then postulate that, just as in Euclidean geometry the separation between points is independent of the coordinate system, the interval as defined above is an *invariant*, so all inertial observers measure the same interval between the same two events. Note that if the events are two points on the trajectory of a light ray, $ds = 0$.

To explore the consequences of this, let us again consider an unprimed frame (t, x, y, z) and a primed frame (t', x', y', z'). Suppose that, as seen in the unprimed frame, the primed frame is moving with speed v in the $+x$ direction (see Figure 1). Also suppose we have set up the axes so that initially the unprimed and primed frames are coincident (i.e., x parallel to x' and so on) and $t = t' = 0$. Our postulate says that

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2dt'^2 + dx'^2 + dy'^2 + dz'^2 \ . \tag{6}$$

We can argue from symmetry that $dy = dy'$ and $dz = dz'$ (**Hint:** consider viewing the same situation from different perspectives, and see if you can arrive at a contradiction if $dy \neq dy'$). Therefore, we are left with

$$-c^2dt^2 + dx^2 = -c^2dt'^2 + dx'^2 \ . \tag{7}$$

We now look for a transformation between the unprimed and primed frames that maintains this invariance. The simplest such transformation turns out to be

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(-\tfrac{v}{c^2}x + t) \ . \end{aligned} \tag{8}$$

This is a *Lorentz transformation*. The generalization to arbitrary directions is straightforward. As before, to change back, we simply flip the sign of v and exchange primed for unprimed variables.

Great, but... what the heck! This says that the measurement of *time* is different in the two frames! This doesn't square with our intuition *at all*. In fact, let's explore some consequences that emerge from this comparison between frames in special relativity.

Consequences

As was discovered well before Einstein proposed special relativity, Maxwell's equations *are* invariant in form under a Lorentz transformation. That's good news. However, there are other implications that may make the cure seem worse than the disease:

Length contraction.—Suppose that in the unprimed frame we measure the length of a stick, oriented along the x axis, that is moving in the primed frame and in that frame has length l. As you may remember from other exposures to special relativity, we have to be precise in how we specify our measurement: in this case, it will be at a single time t as measured in the unprimed frame, meaning $dt = 0$. We then have

$$\begin{aligned} dx' &= \gamma(dx - vdt) \\ l &= \gamma dx \\ dx &= l/\gamma \,. \end{aligned} \tag{9}$$

Noting that $\gamma \geq 1$ for all speeds v, this means that we measure a shorter length in the frame in which the stick is moving. If instead the stick is at rest in the unprimed frame and has length l as measured there, what do we see in the primed frame? The transformation is

$$\begin{aligned} x &= \gamma(x' + vt') \\ \Rightarrow\ dx &= \gamma(dx' + vdt') \end{aligned} \tag{10}$$

hence for $dt' = 0$ we again get that in the frame in which the stick is moving, the length is contracted to l/γ.

Time dilation.—Now suppose that in the unprimed frame we look at a clock in the primed frame. In the primed frame, a time T elapses. How much time goes by in the unprimed frame? For this problem, we note that

$$\begin{aligned} t &= \gamma(\tfrac{v}{c^2}x' + t') \\ \Rightarrow\ dt &= \gamma(\tfrac{v}{c^2}dx' + dt') \,. \end{aligned} \tag{11}$$

If the clock is at rest in the primed frame then $dx' = 0$, so $dt = \gamma dt' = \gamma T$. Therefore, the elapsed time is longer as seen in a frame in which the clock is moving. Note that “clock” here is very general indeed, and refers to anything that takes time. It could be a wristwatch, a chemical process, a nuclear decay, anything at all. At this point, many people like to consider the “twin paradox”: consider identical twins, one of whom stays on Earth and the other of whom blasts off in a rocket, accelerates to nearly the speed of light, travels for a year in her reference frame, then turns around and comes back. The “paradox” is posed as follows: since both twins consider themselves to be at rest, which one should be older when they meet after the journey? The resolution, which I’ll let you ponder, is to determine whether there is any way that you could tell which twin you were. If something breaks the symmetry, one can be older than the other. If not, they have to be the same age.

What does it mean?—The effects discussed above are counterintuitive, to put it mildly. The reason, of course, is that we don’t travel anywhere near the speed of light relative to everyday objects, so we have evolved to be used to Newtonian mechanics. As an example, the fastest speed that most of us have ever traveled is on airplanes, perhaps up to 270 m s^{-1}. The speed of light is about $c = 3 \times 10^8$ m s^{-1}, so the Lorentz factor is $\gamma = 1/\sqrt{1 - v^2/c^2} \approx$

1.00000000000041. That's four parts in 10^{13}! This actually has been detected using atomic clocks flying on planes, but in our everyday life we'd never notice it.

Nonetheless, a lot of people are pretty uncomfortable with the implications of special relativity, which is probably one reason why it is a favorite target of crackpots (another being that Einstein personally was so famous). It is useful to remember that *in the rest frame of something* everything proceeds as normal. Aliens in some distant galaxy might see us appear to move at 90% of the speed of light, but that can't possibly affect us at all. This means, for example, that if I am moving really fast and see a star appear to be contracted by a factor of 10 in my direction of motion, it certainly doesn't imply that there really are huge pressure forces inside the star!

There are, however, real effects than can be and have been measured, and it is this experimental confirmation that gives us confidence in the predictions of special relativity, counterintuitive though they might be. For example, consider a muon, which is a subatomic particle that decays with a characteristic lifetime of $\tau = 2.2 \times 10^{-6}$ seconds. Suppose we set one going at $v/c = 0.9$ of the speed of light. We would expect it to travel a typical distance $D = 0.9c \times 2.2 \times 10^{-6}$ s=590 meters before decaying. Instead, we find that the typical distance is 1360 meters. What is happening? We'll analyze this from two different perspectives:

- From the perspective of the particle, the length of the track on which it is traveling is smaller by a factor of $\gamma = 2.3$. That means that if, in the laboratory frame the track has a length of 1360 meters, in the particle rest frame it appears to have a length of 590 meters, so in the particle rest frame it lasts the expected 2.2 microseconds and all is well.
- From the perspective of the laboratory, the lengths are as expected but the muon decay "clock" runs slowly by a factor of γ and therefore lasts long enough to travel the longer distance.

Therefore, in both frames, observers agree on the final result. This *has* to be the case. In fact, this is a general principle that can help you navigate through tricky situations in relativity. In a given setup, think about facts or numbers on which every observer must agree. Examples might include the number of particles in a box, or whether the muon in the above example reaches the end of a track. These are, if you like, additional examples of invariants: things that are the same in all frames.

Apparent paradoxes of special relativity

Many people are highly uncomfortable with special relativity because at first glance it appears that it predicts things that are blatantly contradictory. Here I offer up for discussion two of the most common.

The barn and ladder paradox.—A very fast runner is handed a 10 meter long ladder and asked to get it inside a barn that is only 8 meters deep. Her assistant, who is stationary relative to the barn, computes that she only needs to run a bit over 60% of the speed of light relative to the barn, because then $1/\gamma = \sqrt{1 - v^2/c^2} = \sqrt{1 - 0.6^2} = 0.8$ and thus the ladder will appear to him to be only 8 meters long. He will then close the barn door after the ladder is completely inside the barn, thus allowing the ladder to be put inside. The runner, however, has read a little more of special relativity. She notes that from her perspective, the *barn* will seem even shorter than it is. She will see it to be only $0.8 \times 8 = 6.4$ meters deep, making it even worse because in her frame the ladder still appears to be 10 meters long.

If they try this experiment, will the ladder ever be completely inside the barn?

The twin paradox.—Pat and Robin are identical twins. On their 20th birthday, Pat climbs aboard a rocket and sets off on a trip that lasts 10 years in Pat's frame. Pat then turns around and takes the trip back to Earth, which also takes 10 years in Pat's frame. Pat's speed on both legs of the trip is 90% of the speed of light. Therefore, Robin reasons that since the Lorentz factor is $\gamma = 1\sqrt{1 - v^2/c^2} = 2.3$, as Robin sees Pat's clock advancing by 20 years over the whole trip, Robin's clock will therefore advance $2.3 \times 20 = 46$ years. Therefore, when they meet, Robin calculates that Pat will be 40 years old and Robin will be 66 years old.

Pat computes things differently. Pat also sees Robin moving at a relative speed of 90% of the speed of light. Therefore, Pat computes that Robin's clock will move a factor of 2.3 times more slowly than Pat's. This means that whereas Pat will age 20 years, Robin will age only $20/2.3 \approx 8.7$ years. Thus when they meet back on Earth, Pat will be 40 years old and Robin will be 28.7 years old.

When the meet, which one will be older?

Summary

Special relativity and much of modern physics (including general relativity and quantum mechanics) operates far beyond our everyday experience. Their predictions work beautifully when tested against experiments and observations, but is this enough? Postmodernists have argued that all scientific theories are socially constructed, and point to the radical change of perspective from Newtonian physics to propose that no theory including our current ones has a greater claim to reality than any other. What do you think?