Newton and the concept of mathematical modeling of physics

Reference webpages:

http://en.wikipedia.org/wiki/Isaac_Newton
http://en.wikipedia.org/wiki/Mathematical_model

In this lecture we will discuss the life and work of the man who is arguably the most influential scientist of all time, and yet who spent most of his time on mystical Biblical interpretation that is considered essentially useless. Here, for the first time, we will bring mathematics into our descriptions. Questions to keep in mind are:

1. What is the relation of Newton's mathematical advances to his physical theories?
2. In terms of mathematical modeling, how important is this in life sciences such as chemistry and biology?
3. Some people have expressed wonderment that mathematical modeling should fit the universe at all, rather than there being chaotic or complicated laws. What do you think?
4. How did Newton's accomplishments change our philosophy?

The life of Newton

Newton was born on Christmas Day 1642 by the old calendar (January 4, 1643 by our current calendar), in the rather remarkably named Woolsthorpe-by-Colsterworth (!!) in England. He was a relatively undistinguished student early on, but when he returned to secondary school in 1660 (after a failed attempt by his mother to make him a farmer!) he became the top-ranked student in the school. Apparently he was motivated at least in part by revenge against a schoolyard bully. His family was poor, but he entered Trinity College, Cambridge in a work-study role, where he had reduced tuition in exchange for the performance of mostly menial tasks. He got his degree from Cambridge in 1665 having been largely an indifferent student (although he did discover the generalized binomial theorem).

Up to this point, no one would have predicted greatness from Newton, but shortly after his graduation the university closed due to the bubonic plague that was then sweeping across London (and that would be terminated when the Great Fire of 1666 destroyed the center parts of the city, reducing crowding and thus making transmission far more difficult). Newton went back to his home in Woolsthorpe, where he embarked on what may have been the most productive two years in scientific history (the only competitor for compressed high productivity is Einstein's "Miracle Year" of 1905, which we'll discuss in a later lecture).

While a student at Cambridge, Newton had engaged in private study of Copernicus, Galileo, and Kepler (in preference to the Aristotelian teachings of the university). This, plus his excursions into mathematics, led him in his two years to (1) establish the principles of calculus, (2) begin his studies of optics, which produced a new theory of color and led to his development of the reflecting telescope, which is the design used by all large telescopes today, (3) begin to establish the principles of mechanics and gravitation. He returned to Cambridge in 1667, and stayed there for the rest of his academic life.

Stimulated (and funded) by his good friend Edmund Halley, Newton layed out his ideas about mechanics, gravity, and sundry other physical subjects in his masterwork "Philosophiæ-Naturalis Principia Mathematica" (known as the Principia, and meaning "Mathematical Principles of Natural Philosophy") in 1687. This is one of the greatest scientific books ever written, and made Newton the most prominent scientist of his time. It also set a standard for physics that we shall explore in this lecture: starting from basic principles that are elucidated mathematically, he derives many propositions, including Kepler's three laws of motion. Many of his derivations are most transparently done using calculus, but because Newton himself had invented calculus and it was therefore not known or used by others, he reduced his demonstrations to geometry (thus paralleling the proof structure seen in Euclid, for example).

Newton eventually became Master of the Mint, where he invented the practice of putting ridges on coins, as this makes counterfeiting and coin shaving more difficult. He became President of the Royal Society, and was the first scientist to be knighted. Most of the last decades of his life, and indeed most of his written output, was spent on alchemy and biblical interpretation that, frankly, strike us as the work of a crackpot; interesting for the greatest scientist of all time. One possible explanation was found after his death: his body had very large amounts of mercury in it, probably from his alchemical experiments, and might explain in part his many eccentricities.

Even without that, however, Newton was not the easiest person to be around. He had a vicious and ongoing battle with Leibniz about who invented calculus (Newton accused Leibniz of plagiarism), and when Robert Hooke criticized some of Newton's ideas about optics this initiated a hostility between the men that persisted until Hooke's death. He also became a bit of a recluse in his later life, although he was still capable of amazing intellectual feats. As an example of this, in 1696 John Bernoulli published a famous mathematical challenge: starting from one point and going to another point below it and to the side, what curve will allow a bead to slide from the first to the second point in the shortest amount of time? Newton, then 53 years old, sent in an anonymous solution that he had obtained on his walk home from work. When the solution arrived, Bernoulli said "I recognize the lion by his paw." Even late in life, Newton was without peer as a mathematician and physicist. His was the physics that was generalized and extended by Einstein, so we will now explore his laws of motion and gravity, using for the first time the mathematical formulations.

Newton's laws of motion

We will explore Newton's laws in three ways. First we will state them qualitatively, i.e., in words. Second, we will introduce Newton's notation of calculus (for those unfamiliar with it) and restate the laws in that way. Third, we will examine what these laws have to say about the conservation of some quantities. That is, we will note that in any closed system (i.e., one that does not interact with anything in the external universe), Newton's laws imply that some characteristics of the system will remain the same no matter how it interacts with itself.

First, the qualitative statement of Newton's laws of motion:

1. An object at rest remains at rest, and an object in uniform straight-line motion remains in that motion, except when acted on by an external force.
2. Force equals mass times acceleration. That is, the force required to accelerate a given mass equals the product of that mass with the acceleration.
3. For every action, there is an equal but opposite reaction.

Now let's explore what this means in more mathematical language. To do this, we need to introduce some notation and definitions.

Vectors.—Vectors are quantities that have both a magnitude and a direction. Velocity is an example. To specify your velocity, you need to indicate your speed and your direction: 60 miles per hour, going north. The speed itself is not a vector, because it is simply a magnitude. Vectors are often represented in boldface, so velocity might be $\mathbf{v}$.

Derivatives.—These come from calculus. It turns out to be extremely common in physics to want to know the rate of change of a quantity over a very small time, or a very small distance, or a very small something else. To express this we have a somewhat unfortunate notation: to indicate a very small amount of change in some quantity x, you put a "d" in front of it: dx. For example, if you want a very short amount of time, and use t to represent that time, then dt is your short interval of time. A small change in velocity would be $d\mathbf{v}$ in the same way; note that a small change in a vector such as velocity is itself a vector, because you have to indicate the change in each of the directions of the vector.

The reason that this is a somewhat unfortunate notation is that we often want to express the change in one quantity with respect to the other, if both changes are small. Therefore, for example, you might want to know the *rate* at which the velocity changes with time. Over a small interval of time dt you have a change of $d\mathbf{v}$ in velocity, so your rate of change in velocity is $d\mathbf{v}/dt$. If you have not seen this notation before, you have a strong tendency to want to cancel the "d"s in this expression to leave $\mathbf{v}/t$, but this is not right.

Velocity.—With that in mind, let us go back to define some quantities. Suppose that a particle is at a position $\mathbf{r}$ that varies with time; the boldface here means that it is a vector, hence we need three numbers to define it (an example triple might be the longitude, latitude, and altitude from the surface of the Earth). Then the velocity is the rate of change of this position with time: $\mathbf{v} = d\mathbf{r}/dt$.

Acceleration.—This is simply the rate of change of velocity with time: $\mathbf{a} = d\mathbf{v}/dt$.

Mass.—Unlike velocity and acceleration, mass is not a vector quantity. It represents the total amount of matter (and energy) in an object. More precisely, it determines the amount of inertia an object has, that is, its resistance to changes in its motion. Mass is represented by m.

Momentum.—In Newtonian mechanics, the momentum is the product of mass and velocity: $\mathbf{p} = m\mathbf{v}$. In Einstein's theory this definition changes as the speed (which, you recall, is the magnitude of the velocity) approaches the speed of light. You can, if you like, think of momentum as an indication of who would win in a collision contest. Imagine two wet balls of clay moving towards each other on a frictionless surface. Assuming that upon collision the two stick together (not bouncing and not having any clay flying out), how does the single ball of clay move after the collision? If, for example, one ball has a mass of 1 kg and is moving at 10 m s^{-1} to the right, and the other has a mass of 2 kg and is moving at 5 m s^{-1} to the left, then since the magnitudes of their momenta are equal but they are moving in opposite directions, after impact they will be stationary.

Force.—Loosely defined, the force $\mathbf{F}$, a vector quantity, indicates how hard is the push or pull on something, and in which direction.

With these definitions, we can restate Newton's laws mathematically:

1. "except when acted on by an external force" means $\mathbf{F} = 0$. "An object at rest remains at rest, and an object in uniform straight-line motion remains in that motion" means that the velocity has not changed, so $d\mathbf{v}/dt = 0$. It turns out, however, to be more profitable to write this in terms of the momentum. In Newtonian mechanics the mass m is constant. Therefore, if $d\mathbf{v}/dt = 0$ then $d(m\mathbf{v})/dt = d\mathbf{p}/dt = 0$ as well. That is, without an external force, the momentum is constant.

2. "Force equals mass times acceleration" translates to $\mathbf{F} = m\mathbf{a}$ in our notation. But recall that $\mathbf{a} = d\mathbf{v}/dt$, so this means $\mathbf{F} = md\mathbf{v}/dt$, or $\mathbf{F} = d(m\mathbf{v})/dt$. Therefore, our completely general rule (which turns out to be valid in Einstein's theory as well if you use the generalized definition of momentum) is

$$d\mathbf{p}/dt = \mathbf{F}\ . \tag{1}$$

Note that this incorporates the first law as well. An auxiliary point is that the force $\mathbf{F}$ can be written as the vector sum of all the forces from every other object, e.g., the force on object A from objects B, C, and D is $\mathbf{F}_{\rm tot} = \mathbf{F}_{BA} + \mathbf{F}_{CA} + \mathbf{F}_{DA}$.

3. "For every action, there is an equal but opposite reaction" is actually a statement about forces. More specifically, consider two objects A and B. Let the (vector!) force of A on B be $\mathbf{F}_{AB}$, and the vector force of B on A be $\mathbf{F}_{BA}$. Then Newton's third law says

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA} \ . \tag{2}$$

Note the amazing generality of this law. It doesn't matter *what* force you consider. It could be gravity, it could be electromagnetism, it could be the strong or weak force, it could be some oddball force no one has ever discovered before. Nonetheless, this symmetry applies. This imposes *tremendous* restrictions on the form a force may take. Another important restriction is that the force between two particles acts along the line between them. Thus, for example, if particle A is at vector location $\mathbf{r}_A$ in some coordinate system, and particle B is at vector location $\mathbf{r}_B$, then the force between them is $\mathbf{F}_{AB} \propto (\mathbf{r}_B - \mathbf{r}_A)$, where the proportionality constant can be positive or negative and depends on the nature of the force.

Perspective and conservation laws

It is useful at this stage to consider how we have traveled to Newton. Early peoples were confronted with a large array of often frightening but seemingly unconnected phenomena. They largely ascribed these to the inscrutable motives of arbitrary and often cruel deities. Patterns were noted in some cases (the regularity of the seasons and when certain crops grow, or the patterns of stars in the heavens). Eventually some people started to measure these patterns, to quantify their regularity in an empirical way. This gradually developed into the statement of particular mathematical expressions that would explain the quantified observations; for example the distance fallen by an object in a vacuum is proportional to the square of the time since it was dropped. Finally, Newton and his contemporaries developed the approach of starting with simple laws and deriving their general consequences, which could then be compared with future observations or experiments. This is the scientific method; we shall explore this much more in the next lecture.

All this is very well, but what consequences do Newton's laws have? In the next section we will explore the specific consequences of his law of gravity, but here we will demonstrate that these laws imply the *conservation* of certain quantities for isolated systems. Why do we care? Imagine some really complicated interaction, e.g., the smash-up of two spaceships in orbit. You've got crushed metal, glass fragments, leaking fuel, the whole bit. You might throw your hands up in despair about the prospect of knowing anything about this collision, but it turns out that there are some quantities that are absolutely the same before the interaction and after. It gives you a strong sense of aesthetic satisfaction, and is incidentally one of the most powerful tools scientists have for predicting the behavior of systems.

To derive one conserved quantity, we will be standard physicists and idealize. Our idealization will be that we have a system that is completely isolated from outside influences. Does this make sense? At first glance, this appears to be absurd; all of us interact with all things, from the air around us to other people to the Sun to the entire universe. Indeed, some Eastern religions (which are enjoying renewed popularity in the West) assert that there is an "energy field" that connects all things, and that one must therefore be holistic in all endeavors. If so, our task is hopeless. What right do we have to treat systems as isolated?

The answer, fundamentally and always, is that experiments show that we can get reproducible and correct predictions for the behavior of systems given a certain amount of isolation. An object that falls in a vacuum tube on Earth will do this in a simple way. An object that slides with minimal friction (e.g., a puck on an air hockey table) does not have influences from a party across campus. From the standpoint of physics, we can often compute the effect of objects that we have ignored, and discover that they are negligible. For example, gravity is universal, yet we don't have to include the effect of the Andromeda galaxy on the orbits of the planets in our solar system. Therefore, in strong distinction to the aesthetic preferences that led astronomers prior to Kepler to assume orbits were based on circles, these simplifications are based on experiment and can be tested.

With that in mind, we will assume the simplest possible system: two point particles, isolated from the rest of the universe, that exert forces on each other.

First, let's consider momentum. We'll name the two particles A and B as before. Let their momenta be $\mathbf{p}_A$ and $\mathbf{p}_B$. How do these momenta change with time? Using the same notation we had before,

$$\begin{aligned} d\mathbf{p}_A/dt &= \mathbf{F}_{BA} \\ d\mathbf{p}_B/dt &= \mathbf{F}_{AB}\ . \end{aligned} \tag{3}$$

But what does this mean about the total momentum $\mathbf{p}_{\rm tot} \equiv \mathbf{p}_A + \mathbf{p}_B$?

$$d\mathbf{p}_{\rm tot}/dt = d\left(\mathbf{p}_A + \mathbf{p}_B\right)/dt = \mathbf{F}_{BA} + \mathbf{F}_{AB} = \mathbf{F}_{BA} - \mathbf{F}_{BA} = 0\ . \tag{4}$$

Therefore, the total momentum is conserved. What if there are more than two particles? Then, since the total force is the linear vector sum of each of the individual forces, one can repeat the procedure above for every pair of particles. The net result is that, again, the total linear momentum is conserved. This is a major result. It doesn't tell us how a system will move or evolve (we need Newton's laws and a force law for that), but it places serious constraints on the states an isolated system can attain.

Another example of a conserved quantity is the *angular momentum*. The most familiar example of this is an ice skater. If she spins slowly with her arms out, then brings her arms in, she spins faster. If you are familiar with cross products, the angular momentum is defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. As with the ordinary (linear) momentum $\mathbf{p}$, angular momentum is conserved for an isolated system; an external torque is needed to change the angular momentum.

To prove this, select an arbitrary point in space, and measure vector positions $\mathbf{r}$ relative to that point. Select a reference frame in which that point is stationary, and in that reference frame determine the momenta $\mathbf{p}$ of particles. We then define the angular momentum $\mathbf{L}$ as

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \ . \tag{5}$$

The time rate of change of the angular momentum is the torque $\mathbf{N}$:

$$\mathbf{N} \equiv d\mathbf{L}/dt = (d\mathbf{r}/dt) \times \mathbf{p} + \mathbf{r} \times (d\mathbf{p}/dt) = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F} \ . \tag{6}$$

How does the angular momentum evolve for an isolated system? Again, let's simplify to a two-particle system. Then the time rate of change of the total angular momentum $\mathbf{L}_{\rm tot} \equiv \mathbf{L}_A + \mathbf{L}_B$ is

$$\begin{aligned} d\mathbf{L}_{\rm tot}/dt &= \mathbf{r}_A \times \mathbf{F}_{BA} + \mathbf{r}_B \times \mathbf{F}_{AB} \\ &= -\mathbf{r}_A \times \mathbf{F}_{AB} + \mathbf{r}_B \times \mathbf{F}_{AB} \\ &= (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F}_{AB} \\ &\propto (\mathbf{r}_B - \mathbf{r}_A) \times (\mathbf{r}_B - \mathbf{r}_A) \\ &= 0 \ . \end{aligned} \tag{7}$$

Therefore, for an isolated two-particle system, the angular momentum is conserved. **Ask class:** what happens for an isolated system with more than two particles? As with linear momentum, the angular momentum is still constant, as can be seen from pairwise cancellation. Thus, *for any isolated system, the linear momentum and angular momentum are constant.* Another way of putting this is that for *any* system, isolated or otherwise, the total linear and angular momentum are changed only by *external* forces and torques, respectively:

$$d\mathbf{p}_{\rm tot}/dt = \mathbf{F}_{\rm ext}, \qquad d\mathbf{L}_{\rm tot}/dt = \mathbf{N}_{\rm ext} \ . \tag{8}$$

Now, remember, these are statements about the *total* linear or angular momentum of an isolated system, *not* the linear or angular momentum of any individual object in that system. As a familiar example, imagine hitting a pool ball with the cue ball on a smooth table. It can be that the cue ball stops, but the target ball moves on. Both have changed their linear momentum, but the total remains the same. Similarly, if you look at the Earth-Moon system, with time the Moon is moving farther away and the Earth's rotation is slowing, but the total angular momentum is constant (well, the Sun exerts an external torque, but this is a small effect).

We will encounter other conservation laws in succeeding lectures, but the point here is that in addition to providing a good quantitative description of a vast array of phenomena with a small number of mathematical laws, Newton's laws also lead to profound insights about what stays the same, not just about what changes.

Newton's universal law of gravity

Let us now consider one particular force law, which is Newton's law of gravity. We will first deal with the magnitude of the force, which is:

$$F = \frac{Gm_1m_2}{r^2} \ . \tag{9}$$

Here F is the magnitude of the force (notice that it is not boldfaced, hence not a vector), G is the universal constant of gravity, m_1 is the mass of the first object, m_2 is the mass of the second object, and r is the distance between them. Gravity is always attractive, so the direction of the force *on* one object is simply the direction *to* the other object. Therefore, the force on object 1 from object 2 is in exactly the opposite direction to the force on object 2 from object 1, but the two forces have identical magnitudes. This therefore satisfies Newton's third law $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$.

Let's take a minute to appreciate what this means. This is a *universal and mutual* law that applies to *any* pair of objects in the universe! This means that laboratory experiments on gravity have bearing on the orbits of stars in distant galaxies. This represents a major philosophical shift. It used to be thought self-evident that the heavens and the Earth were qualitatively different in many respects. However, Newton and subsequent scientists have found that the same basic laws apply everywhere in the universe. As we will discover throughout the class, there are certainly physical realms that we don't normally encounter (e.g., the realm of the very small, the very fast, or the very strongly gravitating). However, the laws are the same.

Newton reached this law via geometrical considerations. A parallel exists in the way that the flux of light diminishes from a source. Since the area of a sphere increases with distance r like r^2, the energy per time per area decreases like r^{-2}. He also observed falling objects on Earth and compared their rates with the rate at which the Moon would have to fall (compensated, of course, by its sideways inertia) to orbit around the Earth. This works well with his law of gravity. Finally, he was able to derive Kepler's three laws of planetary motion using the inverse square law. Kepler's second law (equal areas in equal times) turns out to be even more general; any "central force law", i.e., any law that is directed radially, obeys this law because it is conservation of angular momentum in another guise.

As we discussed above, the success of Newtonian mechanics, and other contemporaneous developments, radically changed the philosophy of the intellectual elite. Now, it appeared, we might have a clockwork universe, where in principle knowledge of the basic laws and initial conditions might allow us to figure out the past, present, and future of all things. Indeed, many philosophers (including fairly recent ones!) misunderstood Newtonian physics to this extent, and thought that quantum mechanics represented an overwhelming shift in predictability. Not so: chaos exists in Newtonian systems, so after a fairly short amount of time even the most precise initial measurements would not suffice for accurate predictions. However, more generally and reasonably, Newton's successes suggested that simple laws might underlie the rich variety of behavior that we see. To quote Alexander Pope: "Nature and Nature's laws lay hid in night: God said, Let Newton be! and all was light."