Astronomy 622 Cosmology Homework Assignment #4 Solutions

1. *(30) Wherever possible, use energy units* $(c=k_B=\hbar=1)$. *Start with assuming that the energy density of the universe is the Planck energy density (Planck energy divided by Planck length cubed) at the Planck time. (Before that, who knows?) So, consider this to be the ultimate initial condition.*

   a. *(10) Solve for the evolution of the energy density as a function of time from the Planck time until t=10*$^{-36}$*s. (Treat this entire period as a radiation dominated era.)*
   At the Planck time $t_{Pl}=5\times10^{-44}$ s, we assume the energy density of the universe is the Planck value $\rho_{initial}=\rho_{Pl}=E_{Pl}/l_{Pl}^3=10^{19}\text{ GeV}/(10^{-35}\text{ m})^3$. Since it's the radiation era, the energy density should evolve as: $\rho\propto a^{-4}$ and $a\propto t^{1/2}\rightarrow\dot{a}/a=1/(2t)$ so the Friedmann equation tells us:
   $$H^2=\frac{1}{4t^2}=\frac{8\pi G}{3}\rho\rightarrow\rho=\frac{3}{32\pi G t^2}\quad\text{or, equivalently,}\quad\rho(t)=\rho_{Pl}\left(\frac{t_{Pl}}{t}\right)^2.$$

   b. *(10) Ignoring inflation, What is the temperature as a function of time based on these values? Reduce it to a function of T, t and numbers (not constants).* Did you get
   $$\frac{T}{1.1\text{ MeV}}=\left(\frac{t}{\text{sec}}\right)^{-1/2}$$
   ? Recall that for a thermal radiation blackbody,
   $\rho=\alpha_{Stefan-Boltzmann}T^4$, so $T(t)=T_{Pl}\left(\frac{t_{Pl}}{t}\right)^{1/2}$. Hey, you could have guessed that based on the fact that $T\sim1/a$ in the first place! So plug in the numbers, and *voilà*!

   c. *(10) Assume that the universe undergoes an inflationary period from t=10*$^{-36}$*s until 10*$^{-34}$*s and insist that the universe has undergone 100 e-foldings in this crazily short time period. What is the value of H during this period? (Hint: can we call it a Hubble constant?)*
   The only thing we don't know in this scenario is how "big" the universe is at the Planck time: $a_{Pl}=a(t_{Pl})$. During the (first?) radiation era, we have $a=a_{Pl}(t/t_{Pl})^{1/2}$ before inflation starts. This radiation era comes to a close at time $t_I=10^{-36}$ s when inflation supposedly starts: $a_I=a_{Pl}(10^{-36}\text{ s}/t_{Pl})^{1/2}\approx10^4 a_{Pl}$.
   Inflation begins a period where the background density, and therefore the Hubble parameter, is constant (like the cosmological constant), so $H^2=H_I^2=(\dot{a}/a)^2=\Lambda_I/3$ where I'm presuming Λ_I isn't the same as today's cosmological constant. This differential equation is dead easy: $da/a=\sqrt{\Lambda_I/3}\,dt\rightarrow a=a_I e^{\sqrt{\Lambda_I/3}(t-t_I)}=a_I e^{H_I(t-t_I)}$.
   Inflation ends at $t_F=10^{-34}$ s. In order to get 100 e-folds of expansion of a from t_I to t_F we have to insist that $H_I(t_F-t_I)\approx H_I t_F=100$, so $H_I^{-1}\approx10^{-36}$ s. In other words, an incredibly fast timescale. (Compare that to today's leisurely, lazy value of $H_0^{-1}=10\text{ Gyr}/h\approx10^{17}$ s.)

2. *(20) Problem A.3.1 in Andrew Liddle's book. See class website for a copy of Appendix A.3.* The answer is in the back. The only thing really missing is the trivial statement that since the particle is relativistic, $p^2=E^2-m^2\approx E^2\approx T^2$. (You may also set k_B = 1) The interaction rate is $\Gamma=n\sigma v=n\sigma p/E\approx n\sigma\propto T^5$ Do you understand why we're comparing the interaction rate with the expansion rate?

3. *(25) Assume that the number densities of both neutrons and protons* $(n_n,\ n_p)$ *follow the Maxwell-Boltzmann distribution when T>1 MeV. (In this problem, mass, temperature and energy are synonymously reported in units of MeV.)*

   a. *(5) Write down the formula for the ratio of the two Maxwell-Boltzmann distributed number densities for two massive species at the same temperature,* $\frac{n_1}{n_2}$ *(don't plug in any numbers yet) which nicely eliminates a host of constants out in front of the exponential. (Also be sure not to bother with* k_B or c^2 *in the exponential!) Now plug in for the proton and neutron masses and simplify your expression to 2 significant figures as a function of T only.* Most of this problem should be in your notes from class: M-B distribution is $n \propto (mT)^{3/2} e^{-m/T}$ so

   $$\frac{n_1}{n_2} = \left(\frac{m_1}{m_2}\right)^{3/2} e^{-(m_1-m_2)/T}.$$

   Since both $m_n = 0.939566\ \text{GeV} \approx m_p = 0.938272\ \text{GeV} \approx 0.94\ \text{GeV}$ to two-digit accuracy, the ratio of masses up front is effectively 1. But the *differences* in the exponential matter! So:

   $$\frac{n_n}{n_p} = e^{-1.29\ \text{MeV}/T}.$$

   b. *(5) As discussed in class, presume that the effective freezeout of neutrinos occurs at T=0.8 MeV (see problem 2 also). What time is it (in seconds) when T=0.8 MeV?*

   $$\frac{T}{1.1\ \text{MeV}} = \left(\frac{t}{\text{sec}}\right)^{-1/2}$$ So $t = 1.9\ \text{s}$, which is usually hand-waved to ~1 s. You may be wondering what inflation does to our simple $T(t)$ equation. Don't. Consider how long inflation lasts and that energy is conserved (but see #4 below).

   c. *(5) Calculate how many neutrons per proton are left at this freezeout.*
   $e^{-1.29/0.8} \approx 0.20$, so 1 in 5.

   d. *(5) The neutrons proceed to be bound up in nuclei, but this takes some time to last because nuclei are just as often broken up by the more energetic tail in the radiation distribution – at least until T=0.06 MeV. What time is it (in seconds) when T=0.06 MeV?* Same calc as part b.: 336 s.

   e. *(5) Assume the neutrons are effectively free and decay until T=0.06 MeV at which point they're bound up entirely in ^{4}He. How many neutrons per proton are left now? What is the predicted mass fraction of baryons in ^{4}He?* Basic decay equation:

   $$\frac{N}{N_0} = e^{-\Delta t/\tau}$$ and $\tau_n = 890\ \text{s}$, so there are 0.69 as many as at the start: so we're down to 1 in 7. Matching two neutrons and two protons to make one ^{4}He leaves us 12 free H, so the fraction of mass tied up as helium is about 1/4$^{\text{th}}$:
   $Y_p = 4\ \text{nucleons}/16\ \text{total nucleons} \approx 0.25$.

4. *(25) The simplest inflaton is a scalar field with a simple potential (and for the record, it's probably wrong).*

a. *(5) The energy density of such a beast is* $\rho=\frac{1}{2}\dot{\phi}^2+V(\phi)$ *and it can be shown that the pressure content of such a field goes like* $p=\frac{1}{2}\dot{\phi}^2-V(\phi)$. *Assume that the universe undergoes a Λ-like period of inflation – what general constraint does this put on what* $\dot{\phi}^2$ *can be compared to* $V(\phi)$? Quite simply, we require:

$$w=\frac{p}{\rho}=\frac{\dot{\phi}^2/2-V(\phi)}{\dot{\phi}^2/2+V(\phi)}\approx-1 \qquad \rightarrow \qquad \dot{\phi}^2\ll V(\phi)$$

b. *(5) Using the fluid energy equation* $\dot{\rho}+3H(p+\rho)=0$ *, plug in the expressions above and show what the fluid energy equation becomes as a function of* $\phi,\dot{\phi}$ and $\ddot{\phi}$. A simple exercise, I hope!?

$$\dot{\rho}+3H(p+\rho)=0$$

$$\ddot{\phi}\dot{\phi}+\frac{dV}{d\phi}\dot{\phi}+3H(\dot{\phi}^2)=0$$

$$\ddot{\phi}+3H\dot{\phi}=-\frac{dV}{d\phi}$$

c. *(5) From your answer to part b., treat the* $\frac{-dV}{d\phi}$ *term as a force and the* $3H\dot{\phi}$ *term as a friction term. In particular, presume that* ϕ *reaches a "terminal velocity." Given the constraint from part a. above, show that we must have:*

$$\left(\frac{dV}{d\phi}\right)^2\ll 9H^2V$$

This is heading towards the idea of "slow roll."

$$\ddot{\phi}=0 \quad \rightarrow \quad \left(\frac{dV}{d\phi}\right)^2=9H^2\dot{\phi}^2\ll 9H^2V$$

d. *(5) But, you know what H has to be during inflation! Rewrite it in terms of* $V(\phi)$ *to show that*

$$\left(\frac{dV}{d\phi}\right)^2\ll 24\pi GV^2 \quad \text{, i.e.,} \quad \left(\frac{E_{Planck}}{V}\frac{dV}{d\phi}\right)^2\ll 1$$

This was dead simple using the Friedmann equation. Ignoring factors like 8π, $E_{\text{Planck}} \sim G^{-1/2}$

e. *(5) Inflation cools the universe absurdly. T still follows our favorite equation* $T\sim\frac{1}{a}$. *Assume the inflaton (whatever* $V(\phi)$ *might be) starts at t=*10^{-36}*s and causes 100 e-folds, what is the temperature at the end of inflation?*
[We know that can't be right, so when inflation ends it has to "reheat" the universe back up to the T it had before by dumping energy into regular matter and energy at the GUT scale (10^{16} *GeV).]*

From your work above, $T(t)=T_{Pl}\left(\frac{t_{Pl}}{t}\right)^{1/2}$ means $T_I=10^{-4}T_{Pl}=10^{15}$ GeV $=10^{28}$ K.

$\frac{a_F}{a_I}=e^{100}\approx 10^{43}$ means that since $T\sim 1/a$ the temperature drops by a factor of 10^{-43} to 10^{-19} eV $=10^{-15}$ K. Chilly!