Lecture 16 : Cosmological Models I

- **Hubble's law and redshift in the new picture**
- **Standard cosmological models - 3 cases**
- **Hubble time and other terminology**
- **The Friedmann equation**
- **The Critical Density and Ω**

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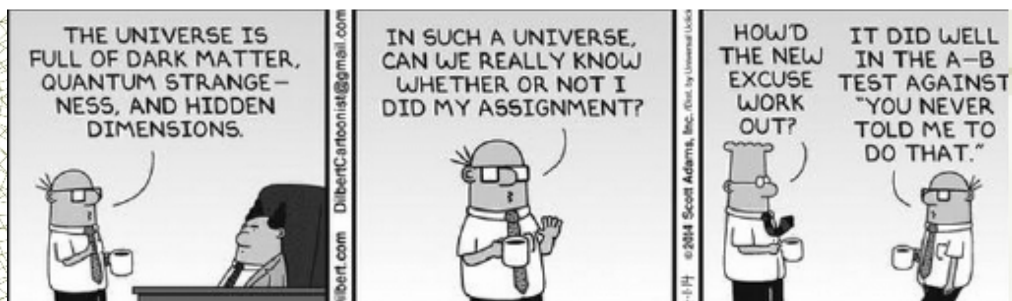
Reading : Chapter 11



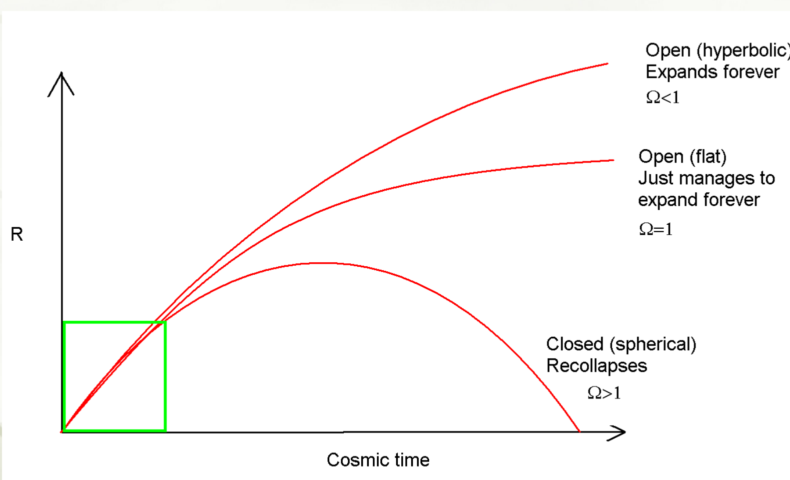
- Homework due Tuesday.
- There is a very nice web page about the 'Big Questions' take a look at http://www.worldscienceu.com/

II : Standard cosmological models- Refresh

- Let's return to question of how scale factor changes over time
  - Equations of GR relates geometry to dynamics (where and how much mass is there in the universe)
  - That means curvature can change with time
  - It turns out that there are three general possibilities for the geometry of the universe and how the scale factor changes with time.

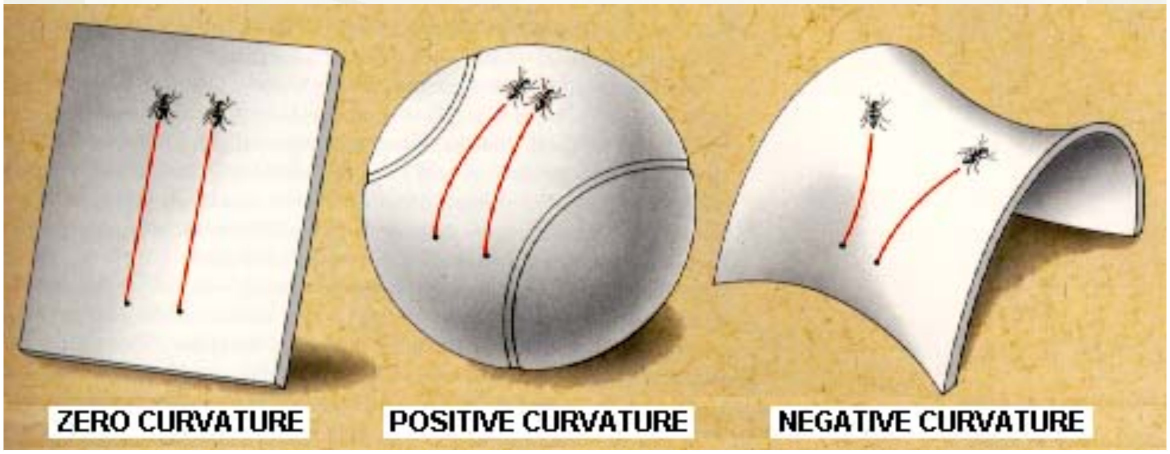
Recap

- Whether case with *k=-1,0, or 1* applies depends on the ratio of the actual density to the "critical" density, Ω
- Properties of standard model solutions:
  - k=-1, Ω<1 expands forever
  - k=0, Ω=1 "just barely" expands forever
  - k=+1, Ω>1 expands to a maximum radius and then recollapses



Curvature of Universe

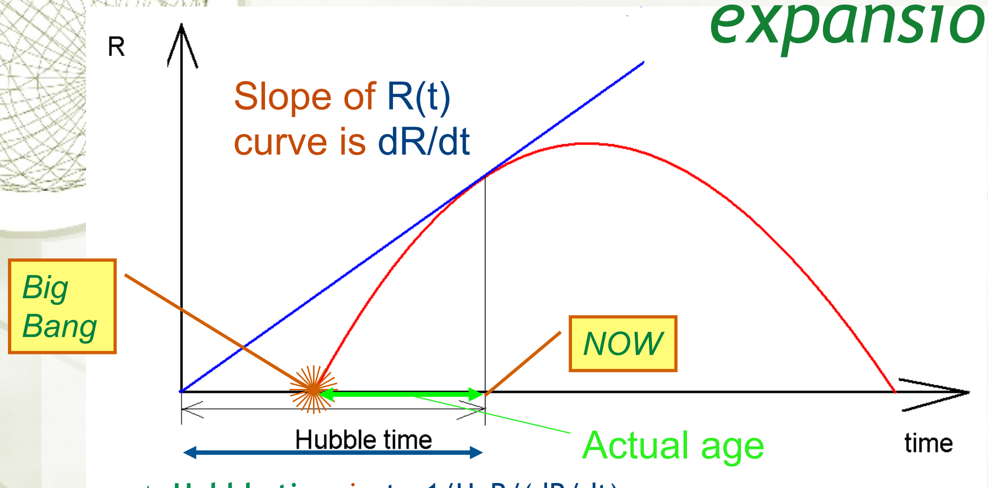
- 3 types of general shapes: flat surface at the left :zero curvature, the spherical surface : positive curvature, and the saddle-shaped surface : negative curvature.
- Each of these possibilities is tied to the amount of mass (and thus to the total strength of gravitation) in the universe, and each implies a different past and future for the universe and how the scale factor changes with time.



Important features of standard models…

- All models begin with R→0 at a <u>finite</u> time in the past
  - This time is known as the **BIG BANG**
  - Space and time come into existence at this moment… **there is no time** before the big bang!
  - The big bang happens everywhere in space… not at a point!
  - Unfortunately our understanding of physics breaks down at very small scales/very early times (Planck scale- 1.6×10^{-35} m at which quantum effects of gravity become strong.)

Hubble time for nonuniform expansion



- **Hubble time** is $t_H=1/H=R/(dR/dt)$
- Since rate of expansion varies, $t_H=1/H$ only gives an estimate of the age of the Universe
- This tends to overestimate the age of the Universe since the Big Bang compared to the actual age

Terminology

- **Hubble distance**, $D=ct_H$ (distance that light travels in a Hubble time). This gives an approximate idea of the size of the observable Universe.
- **Age of the Universe**, t_{age} (the amount of cosmic time since the big bang). In standard models, this is always less than the Hubble time.
- **Look-back time**, t_{lb} (amount of cosmic time that passes between the emission of light by a certain galaxy and the observation of that light by us)
- **Particle horizon** (a sphere centered on the Earth with radius ct_{age}; i.e., the sphere defined by the distance that light can travel since the big bang). This gives the 'edge' of the actual observable Universe.

Hubble time

We can relate this to observations…

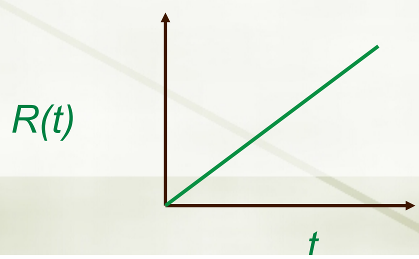
- Once the Hubble parameter has been determined accurately from observations, it gives very useful information about age and size of the expanding Universe…
- Hubble parameter is **ratio of rate of change of size of Universe to size of Universe at a given time**:

$$H = \frac{1}{R}\frac{\Delta R}{\Delta t} = \frac{1}{R}\frac{dR}{dt}$$

- ***If*** Universe ***were*** expanding at a constant rate, we would have

ΔR/Δt=constant and *R(t) =t×(ΔR/Δt)* ; then would have *H=(ΔR/Δt)/R=1/t*

- *ie* $t_H=1/H$ *would be age of Universe since Big Bang*



- There is a connection between the geometry and the dynamics
  - *Closed* solutions for universe expand to maximum size then re-collapse
  - *Open* solutions for universe expand forever
  - *Flat* solution for universe expands forever (but only just barely… almost grinds to a halt).

III : The Friedman equation (or "let's get a bit technical"!- pgs 320-325 in text)

- When we go through the GR stuff, we get the **Friedmann Equation…** this is what determines the dynamics of the Universe

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

- "k" is the curvature constant…
  - k=+1 for spherical case
  - k=0 for flat case
  - k=-1 for hyperbolic case
- ρ is the density of matter (on average) in the universe- changes with time as the universe expands (matter+energy is not created or destroyed, conservation of energy)

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

- What are the terms involved?
  - G is Newton's universal constant of gravitation
  - *dR/dt* is the rate of change of the cosmic scale factor

    This is same as *ΔR/Δt* for small changes in time

    In textbook, symbol for *dR/dt* is $\dot{R}$ *(pronounced "R-dot")*
  - ρ is the total energy density $\div c^2$; this equals mass/volume for "matter-dominated" Universe
  - *k* is the geometric curvature constant (= *+1,0,-1)*

III : The Friedman equation (or "let's get a bit technical!- pgs 320-325 in text)

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

- left hand side- (see eq 11.3,11.4 in text) d^2R/dt^2 corresponds to $\Delta x/\Delta t$ ('v' velocity) squared ; if we think Newtonian for a moment this can be related to energy= $1/2mv^2$

The Friedman Eq - cont

- The text uses the analogy of an 'escape' velocity for the expanding universe
- That is: I can throw a ball off the earth that can escape to infinity, fall back to earth, or just barely escape
- One can also think in terms of the amount of energy required to get to this velocity

- In Newtonian physics one can calculate all this more or less exactly and using the Friedman 'R' one gets
- $d^2R/dt^2 = (4/3\pi)G\rho R$- where ρ is the mass-energy density and
- $(dR/dt)^2 = (8/3\pi)G\rho R^2$ + energy
  - This 2nd equation is the analogy to the Friedman eq

Obtaining Friedmann eq. for mass-only Universe from Newtonian theory

- Consider spherical piece of the Universe large enough to contain many galaxies, but much smaller than the Hubble radius (distance light travels in a Hubble time).
  - Radius is r. Mass is $M(r)=4\pi r^3\rho/3$
- Consider particle m at edge of this sphere; it feels gravitational force from interior of sphere, $F=-GM(r)m/r^2$
- Suppose outer edge, including m, is expanding at a speed

  $v(r)= \Delta r/\Delta t=dr/dt$
- Then, from Newton's 2nd law, rate of change of v is the acceleration $a =\Delta v/\Delta t=dv/dt$, *with* $F=ma$, yielding

  $a= F/m =-GM(r)/r^2 =-4\pi G r\rho/3$
- Using calculus, this can be worked on to obtain

  $v^2 =8/3\pi G\rho r^2$ + *constant*

- But if we identify the "constant" (which is twice the Newtonian energy) with $-kc^2$, and reinterpret r as R, the cosmic scale factor, we have Friedmann's equation!

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

- Divide the Friedman equation by R^2 and we get…

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2}$$

How did I get that??

H=Hubble's constant-Hubble law v=Hd (v= velocity, d= distance of the object)

v=Δd/Δt (we substitute the scale factor R for d)

H= v/d=(dR/dt)/R - so =$(dR/dt)^2 = (H*R)^2$

Now divide by R^2

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

- Divide the Friedman equation by R^2 and we get…

$$H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2}$$

- Let's examine this equation…
- H^2 must be positive… so the RHS of this equation must also be **positive.**
- Suppose density is *zero* (ρ=0)
  - Then, we **must** have negative k (i.e., k=-1)
  - So, empty universes are open and expand forever
  - Flat and spherical Universes can only occur in presence of (enough) matter.

- Now, suppose the Universe is flat (k=0)
  - Friedmann equation then gives

$$H^2 = \frac{8\pi G}{3}\rho$$

  - It turns out that the universe *is* exactly flat *if* the density, ρ, is exactly equal to **the critical density…**(good at all times)

$$\rho = \rho_{crit} = \frac{3H^2}{8\pi G}$$

Value of critical density

- For present best-observed value of the Hubble constant, H_0=68 km/s/Mpc

critical density, $\rho_{critical}=3H_0^2/(8\pi G)$, is equal to

$\rho_{critical}=10^{-26}$ kg/m^3 ; i.e. *6 H atoms/m^3*

- Compare to:
  - ρ_{water} = 1000 kg/m^3
  - ρ_{air}=1.25 kg/m^3 (at sea level)
  - $\rho_{interstellar\ gas}$ =2 × 10^{-21} kg/m^3 (in our galaxy)

  - Universe is a pretty empty place!

- In general case, we can define the **density parameter, Ω,…**

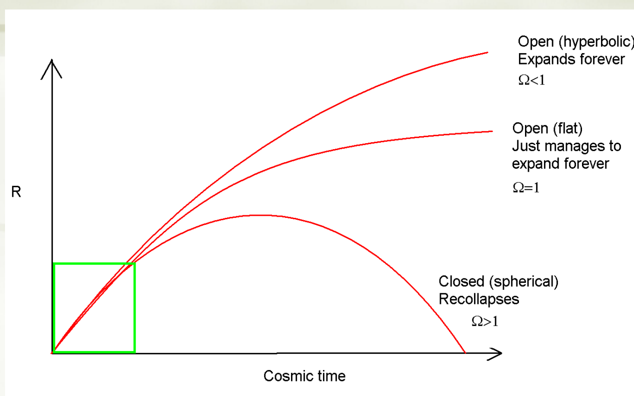
$$\Omega = \frac{\rho}{\rho_c}$$

- Can now rewrite Friedmann's equation yet again using this… we get

$$\Omega = 1 + \frac{kc^2}{H^2 R^2}$$

Omega in standard models

$$\Omega = 1 + \frac{kc^2}{H^2 R^2}$$



- Thus, within context of the standard model:
  - Ω<1 k=-1; then universe is hyperbolic and will expand forever
  - Ω=1 k=0; then universe is flat and will (just manage to) expand forever
  - Ω>1 k=+1; then universe is spherical and will recollapse
- Physical interpretation:
  If there is more than a certain amount of matter in the universe ($\rho > \rho_{critical}$), the attractive nature of gravity will ensure that the Universe recollapses (eventually)!

Review

- The **Friedmann equation** is obtained by plugging each of the possible FRW metric cases into Einstein's GR equation
- Result is an equation saying how the cosmic scale factor *R(t)* changes with time:

$$\left(\frac{dR}{dt}\right)^2 = H^2R^2 = \frac{8\pi G}{3}\rho R^2 - kc^2$$

- The Friedmann equation can also be written as:

$$\Omega = 1 + \frac{kc^2}{H^2R^2}$$

- where

$$\Omega \equiv \frac{\rho}{\rho_{crit}} \equiv \frac{\rho}{(3H^2/8\pi G)}$$

(ok… my brain hurts… what do I need to remember from the past 4 slides???)

- Important take-home message… within context of the "standard model":
  - Ω<1 means universe is hyperbolic and will expand forever
  - Ω=1 means universe is flat and will (just manage to) expand forever
  - Ω>1 means universe is spherical and will recollapse
- Physical interpretation… if there is more than a certain density (on average) of matter in the universe, the attractive nature of gravity will ensure that the Universe recollapses.

III : SOME USEFUL DEFINITIONS

- Have already come across…
  - **Standard model** (Homogeneous & Isotropic GR-based models, **ignoring Dark Energy**!!)
  - **Critical density** ρ_c (average density needed to just make the Universe flat)
  - **Density parameter** $\Omega=\rho/\rho_c$
- We will also define…
  - **Cosmic time** (time as measured by a clock which is stationary in co-moving coordinates, i.e., stationary with respect to the expanding Universe)

- **Hubble distance,** $D=ct_H$ (distance that light travels in a Hubble time). This gives an approximate idea of the size of the observable Universe.
- **Age of the Universe**, t_{age} (the amount of cosmic time since the big bang). In standard models, this is always less than the Hubble time.
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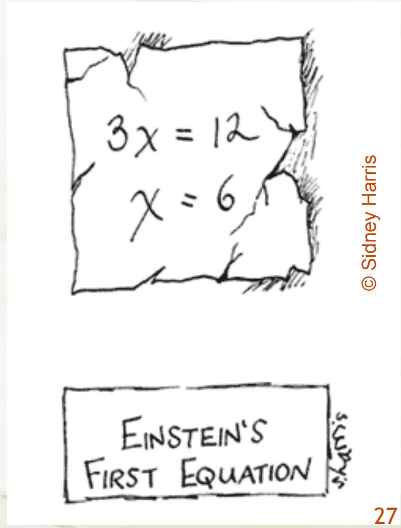
Cosmological Models II

- Deceleration parameter
- Beyond standard cosmological models
  - The Friedman equation with Λ
  - Effects of nonzero Λ
- Solutions for special cases
  - de Sitter solution
  - Static model
  - Steady state model



The deceleration parameter, *q*

- The **deceleration parameter** measures how quickly the universe is decelerating (or accelerating), i.e. *how much the R(t)* graph curves
- In standard models, deceleration occurs because the gravity of matter slows the rate of expansion

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- For those comfortable with calculus, actual definition of *q* is:

$$q = -\frac{1}{RH^2}\frac{d^2R}{dt^2}$$

- In the textbook, d^2R/dt^2 is written as $\ddot{R}$, pronounced "R double-dot"

Matter-only standard model

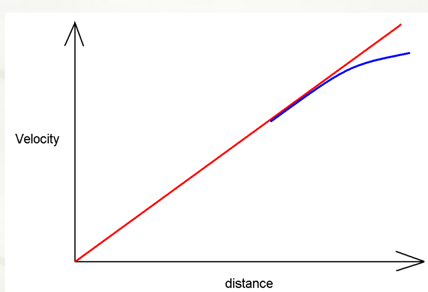
- In standard models where density ρ is entirely from the rest mass energy of matter (ignoring radiation), it turns out that the value of the deceleration parameter is given by

$$q = \frac{\Omega}{2}$$

- We can attempt to measure Ω in two ways:
  - Direct measurement of how much mass is in the Universe --i.e. measure mass density ρ , measure Hubble parameter *H*, and then compare ρ to the critical value $\rho_{crit}=3H^2/(8\pi G)$
  - Use measurement of deceleration parameter, *q- by measuring how much R(t) curves with time (distance from us)*
  - Measurement of *q* is analogous to measurement of Hubble parameter, by observing change in expansion rate as a function of time: need to look at how *H changes with redshift* for distant galaxies

Direct observation of *q*

- Deceleration shows up as a deviation from Hubble's law…



- A very subtle effect – have to detect deviations from Hubble's law for objects with a large redshift

- Newtonian interpretation is therefore:
  - k=-1 is “positive energy” universe (which is why it expands forever)
  - k=+1 is “negative energy” universe (which is why it recollapses at finite time)
  - k=0 is “zero energy” universe (which is why it expands forever but slowly grinds to a halt at infinite time)
- Analogies in terms of throwing a ball in the air…

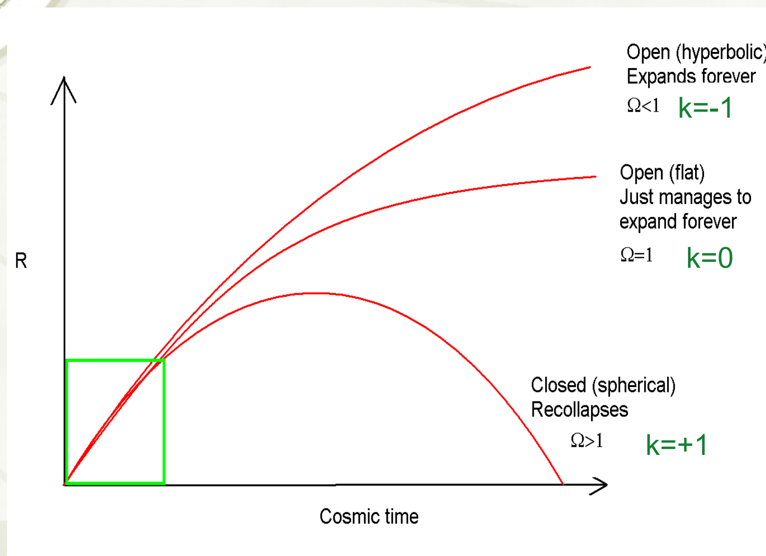
Expansion rates

- For flat (k=0, Ω=1), matter-dominated universe, it turns out there is a simple solution to how *R* varies with *t* :

$$R(t) = R(t_0)\left(\frac{t}{t_0}\right)^{2/3}$$

  - This is known as the Einstein-de Sitter solution
  - For this solution, $V=\Delta R/\Delta t=(2/3)(R(t_0)/t_0)(t/t_0)^{-1/3}$
  - How does this behave for large time? What is $H=V/R$?
- In solutions with Ω>1, expansion is slower (followed by recollapse)
- In solutions with Ω<1, expansion is faster

✦ Open, flat, and closed solutions result for different values of Ω



Modified Einstein's equation

- But Einstein's equations most generally also can include an extra constant term; i.e. in

$$\underline{\underline{\mathbf{G}}} = \frac{8\pi G}{c^4}\underline{\underline{\mathbf{T}}}$$

the **T** term has an additional term which just depends on space-time geometry times a constant factor, Λ

- This constant Λ (Greek letter "Lambda") is known as the "***cosmological constant***";
- Λ corresponds to a "vacuum energy", i.e. an energy **not** associated with either matter or radiation
- Λ could be positive or negative
  - Positive Λ would act as a repulsive force which tends to make Universe expand faster
  - Negative Λ would act as an attractive force which tends to make Universe expand slower
- Energy terms in cosmology arising from positive Λ are now often referred to as "***dark energy***"

We really do not know what this is or means.....

Modified Friedmann Equation

- When Einstein equation is modified to include Λ, the Friedmann equation governing evolution of *R(t)* changes to become:

$$\left(\frac{dR}{dt}\right)^2 = H^2R^2 = \frac{8\pi G}{3}\rho R^2 + \frac{\Lambda R^2}{3} - kc^2$$

- Dividing by $(HR)^2$, we can consider the relative contributions of the various terms evaluated at the present time, t_0
- The term from matter at t_0 has subscript "M";
- Two additional "Ω" density parameter terms at t_0 are defined:

$$\Omega_M \equiv \frac{\rho_0}{\rho_{crit}} \equiv \frac{\rho_0}{(3H_0^{\;2}/8\pi G)} \qquad \Omega_\Lambda \equiv \frac{\Lambda}{3H_0^{\;2}} \qquad \Omega_k \equiv -\frac{kc^2}{R_0^{\;2}H_0^{\;2}}$$

- Altogether, at the present time, t_0, we have

$$\Omega total = \Omega_M + \Omega_\Lambda + \Omega_k$$

Generalized Friedmann Equation in terms of Ω's

- The generalized Friedmann equation governing evolution of *R(t)* is written in terms of the present Ω's (density parameter terms) as:

$$\dot{R}^2 = \left(\frac{dR}{dt}\right)^2 = H^2R^2 = H_0^{\;2}R_0^{\;2}\left[\Omega_M\left(\frac{R_0}{R}\right) + \Omega_\Lambda\left(\frac{R}{R_0}\right)^2 + \Omega_k\right]$$

- The **only** terms in this equation that vary with time are the scale factor *R* and its rate of change *dR/dt*
- Once the constants H_0, Ω_M, Ω_Λ, Ω_k are measured empirically (using observations), then whole future of the Universe is determined by solving this equation!-A major activity of astronomers today
- Solutions, however, are more complicated than when Λ=0 …

Effects of Λ

- Deceleration parameter (observable) now depends on both matter content and Λ (will discuss more later)
- This changes the relation between evolution and geometry. Depending on value of Λ,
  - closed (k=+1) Universe could expand forever
  - flat (k=0) or hyperbolic (k=-1) Universe could recollapse
  - Rate of change of scale factor has complex behavior with time

"asymptotic" behavior

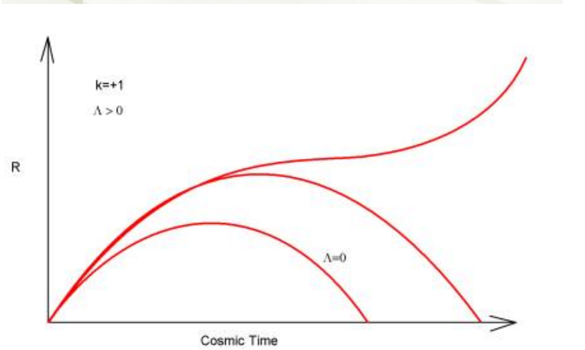
$$\dot{R}^2 = \left(\frac{dR}{dt}\right)^2 = H^2R^2 = H_0{}^2R_0{}^2\left[\Omega_M\left(\frac{R_0}{R}\right) + \Omega_\Lambda\left(\frac{R}{R_0}\right)^2 + \Omega_k\right]$$

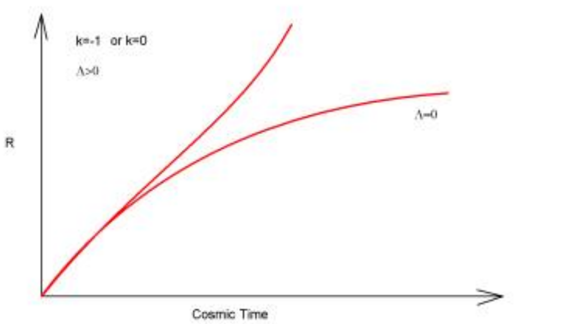
- Different terms in modified Friedmann equation are important at different times…
  - Early times ⇒ *R* is small
  - Late times ⇒ *R* is large
- When can curvature term be neglected?
- When can lambda term be neglected?
- When can matter term be neglected?
- How does *R* depend on *t* at early times in ***all*** solutions?
- How does *R* depend on *t* at late times in ***all*** solutions?
- What is the ultimate fate of the Universe if $\Lambda \neq 0$?

Consequences of positive Λ

- Because Λ term appears with *positive* power of *R* in Friedmann equation, effects of Λ **increase** with time if *R* keeps increasing
- Positive Λ can create accelerating expansion!

$$\dot{R}^2 = \left(\frac{dR}{dt}\right)^2 = H^2R^2 = H_0^{\,2}R_0^{\,2}\left[\Omega_M\left(\frac{R_0}{R}\right) + \Omega_\Lambda\left(\frac{R}{R_0}\right)^2 + \cancel{\Omega_k}\right]$$





Other ideas

Steady state universe ?
Constant expansion rate
Matter constantly created
No Big Bang

All Ruled out by existing observations:

Distant galaxies (seen as they were light travel time in the past) differ from modern galaxies

Cosmic microwave background implies earlier state with uniform hot conditions (big bang)

Observed deceleration parameter differs from what would be required for steady model

Next lecture

The Early Universe
- Cosmic radiation and matter densities
- The hot big bang
- Fundamental particles and forces
- Stages of evolution in the early Universe

Next lecture…

- The Early Universe
  - Cosmic radiation and matter densities
  - The hot big bang
  - Fundamental particles and forces
  - Stages of evolution in the early Universe

Special solutions

- "de Sitter" model:
  - Case with Ω_k=0 (flat space!), Ω_M =0 (no matter!), and Λ >0
  - Then modified Friedmann equation reduces to

$$\dot{R}^2 = H^2 R^2 = H_0{}^2 R_0{}^2 \Omega_\Lambda \left(\frac{R}{R_0}\right)^2 = \frac{R^2 \Lambda}{3}$$

  - Hubble parameter is constant:

$$H = \frac{\dot{R}}{R} = \sqrt{\frac{\Lambda}{3}}$$

  - Expansion is exponential:

$$R = R_0 e^{Ht/t_0}$$