A Guide to the Next Few Lectures

•The geometry of gravitational potentials : methods to derive gravitational potentials from mass distributions, and visa versa.

•Potentials define how stars move

•The gravitational field and stellar motion are interconnected : the Virial Theorem relates the global potential energy and kinetic energy of the system.

• Collisions?
• The Distribution Function (DF) :
the DF specifies how stars are distributed throughout the system and with what velocities.

For collisionless systems, the DF is constrained by a continuity equation : the Collisionless Boltzmann Equation

•This can be recast in more observational terms as the Jeans Equation.
The Jeans Theorem helps us choose DFs which are solutions to the continuity equations

A Reminder of Newtonian Physics sec 3.1 in S&G

Newtons law of gravity tells us that two masses attract each other with a force

eq 3.1 $$\frac{d}{dt}(m\mathbf{v}) = -\frac{GmM}{r^3}\mathbf{r}$$

ϕ (x) is the potential

If we have a collection of masses acting on a mass m_α the force is

$$\frac{d}{dt}(m_\alpha \mathbf{v}_\alpha) = -\sum_\beta \frac{Gm_\alpha M_\beta}{|\mathbf{x}_\alpha - \mathbf{x}_\beta|^3}(\mathbf{x}_\alpha - \mathbf{x}_\beta),\ \alpha \neq \beta$$ eq 3.2

eq 3.3 $$\frac{d}{dt}(m\mathbf{v}) = -m\nabla\Phi(\mathbf{x}),$$

with

eq 3.4 $$\Phi(\mathbf{x}) = -\sum_\alpha \frac{Gm_\alpha}{|\mathbf{x}-\mathbf{x}_\alpha|},\ for\ \mathbf{x} \neq \mathbf{x}_\alpha$$

Gauss's thm $\int \nabla\phi \bullet ds^2 = 4\pi GM$
the Integral of the normal component over a closed surface =$4\pi G$ x mass within that surface

the gravitational potential. If we can approximate the discrete stellar distribution with a continuous distribution ρ.

$$\Phi(\mathbf{x}) = -\int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'$$

ρ(x) is the mass density distribution

Conservation of Energy and Angular Momentum

In the absence of external forces a star will conserve energy along its orbit

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v}) = -m\,\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}),$$

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v}) + m\,\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}) = 0$$

But since $\frac{d\Phi}{dt} = \boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x})$

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\frac{d}{dt}\left[\frac{m}{2}(v^2) + m\Phi(\boldsymbol{x})\right] = 0$$

This is just the KE + PE

$$\frac{dL}{dt} = \boldsymbol{x}\times m\frac{d\,\boldsymbol{v}}{dt} = -m\,\boldsymbol{x}\times\nabla\Phi$$

Angular momentum L

where $\{\hat{x}, \hat{y}, \hat{z}\}$ are the unit vectors in their respective directions.

Some Basics - M. Whittle

- The gravitational potential energy is a scalar field
- its gradient gives the net gravitational force (per unit mass) which is a vector field : see S&G pg 113

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla\cdot\nabla = \nabla^2$$

$$\Phi(\mathbf{r}) = -G\int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}|}\,d^3\mathbf{r}'$$

$$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = G\int_V \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}'-\mathbf{r}|^3}\,\rho(\mathbf{r}')\,d^3\mathbf{r}'$$

$$\nabla\cdot\mathbf{F}(\mathbf{r}) = -4\pi G\rho(\mathbf{r})$$

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})$$ ⟷ Poissons eq inside the mass distribution

$$\nabla^2\Phi(\mathbf{r}) = 0$$ ⟷ Outside the mass dist

Poisson's Eq+ Definition of Potential Energy (W)

So the force per unit mass is

ρ(x) is the density dist

$$\boldsymbol{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = \int G\rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}|^3}d^3\mathbf{x}'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides

S+G pg 112-113

$$\nabla^2\Phi(\mathbf{x}) = -\nabla^2\int\frac{G\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'$$

$$= 4\pi G\rho(\mathbf{x})$$ Poisson's equation.

Potential energy W

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\,\Phi(\mathbf{r})\,d^3\mathbf{r} = -\frac{1}{8\pi G}\int_V |\nabla\Phi|^2\,d^3\mathbf{r}$$

Derivation of Poisson's Eq

So the force per unit mass is

$$\boldsymbol{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = \int G\rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}|^3}d^3\mathbf{x}'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides

$$\nabla^2\Phi(\mathbf{x}) = -\nabla^2\int\frac{G\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'$$

$$= 4\pi G\rho(\mathbf{x})$$ Poisson's equation.

see S+G pg112 for detailed derivation or web page 'Poisson's equation'

More Newton-Spherical Systems

Newtons 1st theorem: a body inside a spherical shell has no net gravitational force from that shell; e.g. $\nabla\Phi(r)=0$

Newtons 2nd theorem: the gravitational force on a body outside a spherical shell is the same as if all the mass were at a point at the center of the shell.

Simple examples:
Point source of mass M; potential $\Phi(r) = -GM/r$;
definition of circular speed; speed of a test particle on a circular orbit at radius r
$v^2_{circular}=r\, d\Phi(r)/dr = GM/r$; $v_{circular}=\mathrm{sqrt}(GM/r)$;Keplerian

escape speed $=\mathrm{sqrt}[2\Phi(r)]=\mathrm{sqrt}(2GM/r)$; from equating kinetic energy to potential energy $1/2mv^2=|\Phi(r)|$

Characteristic Velocities

$v^2_{circular}=r\, d\Phi(r)/dr = GM/r$; $v=\mathrm{sqrt}(GM/r)$:Keplerian

velocity dispersion σ : statistical dispersion of velocities about the mean velocity; if $f(v)dv$ is the velocity field

- $\sigma = [(v - v)^2 f(v)dv]^{1/2}$.

For most systems $\sigma \sim (GM/r)^{1/2}$

escape speed $=v_{esc}=\mathrm{sqrt}(2\Phi(r))$ or $\Phi(r)=1/2v^2_{esc}$

Escape Speed

- As r goes to infinity $\phi(r)$ goes to zero
- so to escape $v^2>2\phi(r)$; e.q. $v_{esc}=sqrt(-2\phi(r))$

- Alternate derivation using conservation of energy
- Kinetic + Gravitational Potential energy is constant

$$KE_1+U_1=KE_2+U_2$$

- Grav potential =-GMm/r; $KE=1/2mv_{escape}^2$
- Since final velocity=0 (just escapes) and U at infinity=0
- $1/2mv_{escape}^2-GMm/r=0$

- The star's *energy E* is the sum of its *kinetic energy KE* = $m\mathbf{v}^2/2$ and the *potential energy PE* = $m\phi(\mathbf{x})$.
- The kinetic energy cannot be negative, and since far from an isolated galaxy or star cluster, $\phi(\mathbf{x}) \to 0$.
- So a star at position **x** can escape only if it has $E > 0$; it must be moving faster than the local

escape speed v_e, given by

$$v^2_e = -2\phi(\mathbf{x}).$$

Gravity and Dynamics-Spherical Systems- Repeat

- Newtons 1st theorm : a body inside a a spherical shell has no net force from that shell $\nabla\phi = 0$
- Newtons 2nd theorm ; a body outside the shell experiences forces as if they all came from a point at the center of the shell-Gravitational force at a point outside a closed **sphere** is the same as if all the mass were at the center
  - This does not work for a thin disk- cannot ignore what is outside of a given radius
- One of the prime observables (especially for spirals) is the **circular velocity;** in general it is $V_c^2(R)/R=G(M<R)/R^2$; more accurate estimates need to know shape of potential
- **so one can derive the mass of a flattened system from the rotation curve**

- point source has a potential $\phi=-GM/r$
- A body in orbit around this point mass has a circular speed $v_c^2=r\, \phi d/dr=GM/r$
- v_c=sqrt(GM/r); Keplerian
- Escape speed from this potential v_{escape}=sqrt(2ϕ)=sqrt(2GM/r) (conservation of energy KE=1/2mv^2_{escape})

Variety of "Simple" Potentials See problems 3.1-3.4,3.7 in S&G

- Point mass $\phi(r)=-GM/r$
- Plummer sphere : simple model for spherical systems
  - $\phi(r)=-GM/\mathrm{sqrt}(r^2+a^2)$
- Singular isothermal sphere $\phi(r)=4\pi G r^2_0 \rho(r_0) \ln (r/r_0)$
  some interesting properties- circular speed is constant at sqrt $(4\pi G r^2_0 \rho(r_0))$
- A disk $\phi(R, z) = - GM/sqrt(R^2 + (a_K + |z|)^2$
- The *Navarro–Frenk–White* (NFW) potential
  $\phi(R)=4\pi G a^2_0 \rho(r_0)$ [ln(1+r/a_0)/(r/a_0)
  – this form fits numerical simulations of structure growth

Homogenous Sphere B&T sec 2.2.2

- Constant density sphere of radius a and density ρ_0

- $M(r)=4\pi Gr^3\rho_0$; r<a
- $M(r)=4\pi Ga^3\rho_0$; r>a

$\phi(R)=-d/dr(M(R))$

R>a: $\phi(r)=4\pi Ga^3\rho_0=-GM/r$

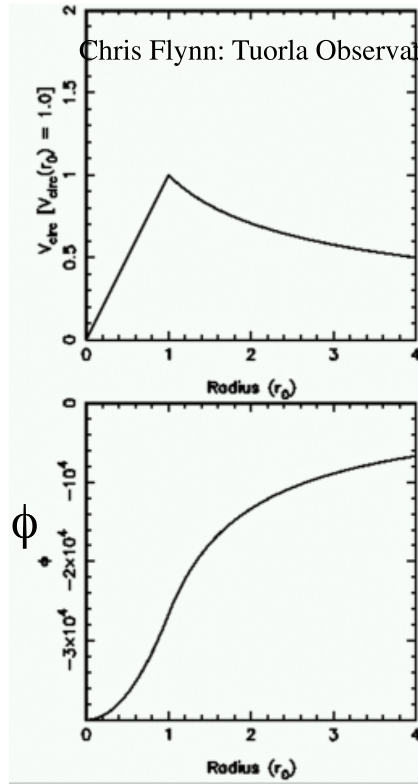
R<a: $\phi(r)=-2\pi G\rho_0(a^2-1/3r^2)$);

$v^2_{circ}=(4\pi/3)G\rho_0 r^2$; solid body rotation R<a

Chris Flynn: Tuorla Observatory

v_c

ϕ



Some Simple Cases

- Constant density sphere of radius a and density ρ_0

Potential energy (B&T) eq 2.41, 2.32

$\phi(R)=-d/dr(M(R))$;

R>a: $\phi(r)=4\pi Ga^3\rho_0=-GM/r$

R<a : $\phi(r)=-2\pi G\rho_0(a^2-1/3r^2)$);

$v^2_{circ}=(4\pi/3)G\rho_0 r^2$ solid body rotation

Potential is the same form as a harmonic oscillator

e.g. the eq of motion is $d^2r/dt^2=-GM(r)/r=4\pi/3Gr\rho$; solution to harmonic oscillator is

$r=A\cos(\omega t+\phi)$ with $\omega=\mathrm{sqrt}(4\pi/3G\rho)=2\pi/T$

$T=\mathrm{sqrt}(3\pi/G\rho_0)=2\pi r/v_{circ}$

Homogenous Sphere

- Potential energy of a self gravitating sphere of constant density r, mass M and radius R is obtained by integrating the gravitational potential over the whole sphere
- **Potential energy $U=1/2\int r\rho(r)\nabla\Phi d^3r$**

$U = \int_0^R -4\pi G\, M(r)\, \varrho(r)\, r\, dr = \int_0^R G[(4/3\pi\rho r^3) x\, (4\pi\rho r^2) dr]/r$

$=(16/3)\pi^2\rho^2 r^2)\int_0^R r^4 dr == (16/15)\pi^2\rho^2 R^5$

using the definition of total mass M (volume x density)

$M=(4/3)\pi\rho R^3$

gives $U=-\ (3/5)GM^2/R$

Homogenous Sphere B&T sec 2.2.2

Orbital period $\mathbf{T=2\pi r/v_{circ}=sqrt(3\pi/G\rho_0)}$

Dynamical time=crossing time

$=T/4=\mathbf{sqrt(3\pi/16G\rho_0)}$

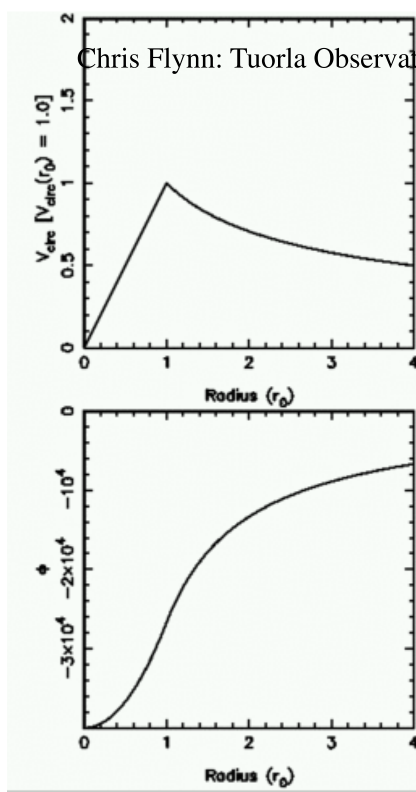
Potential is the same form as an harmonic oscillator with angular freq $2\pi/T$ (B&T 2.2.2(b))

Regardless of initial value of distance (r) a particle will reach r=0 (in free fall) in a time T=/4

Eq of motion of a test particle INSIDE the sphere is

$dr^2/dt^2=-GM(r)/r^2=-(4\pi/3)G\rho_0 r$

General result dynamical time ~sqrt(1/Gρ)



Spherical Systems: Homogenous sphere of radius a

Summary

- $M(r)=4/3\pi r^3\rho$ (r<a); r>a $M(r)=4/3\pi r^3 a$
- Inside body (r<a); $\phi(r)=-2\pi G\rho(a^2-1/3\ r^2)$ (from eq. 2.38 in B&T)

Outside (r>a);)$\phi(r)= -4\pi G\rho(a^3/3)$

Solid body rotation $v_c{}^2 = -4\pi G\rho(r^2/3)$

Orbital period $T=2\pi r/v_c$=sqrt($3\pi/G\rho$);

a crossing time (dynamical time) =T/4=sqrt($3\pi/16G\rho$)

potential energy U$=-3/5GM^2/a$

The motion of a test particle inside this sphere is that of a simple harmonic oscillator $d^2r/dt^2= -G(M(r)/\ r^2=4\pi G\rho r/3$ with angular freq $2\pi/T$

no matter the initial value of r, a particle will reach r=0 in the dynamical time $T/4$

In general the dynamical time $t_{dyn}\sim 1/\mathrm{sqrt}(G<\rho>)$

and its 'gravitational radius' $r_g= GM^2/\mathbf{U}$

Star Motions in a Simple Potential

- if the density ϱ in a spherical galaxy is constant, then a star following a circular orbit moves so that its angular speed $\Omega(r)=V(r\ )/r$ is constant.
- a star moving on a radial orbit, i.e., in a straight line through the center, would oscillate harmonically in radius with period

  P =sqrt[$3\pi/G\varrho$]~ $3t_{ff,}$ where t_{ff} =sqrt[$1/G\varrho$]: S&G sec 3.1

ϱ is the density

Not so Simple - Plummer Potential (Problem 3.2S&G)

- Many astrophysical systems have a 'core'; e.g. the surface brightness flattens in the center (globular clusters, elliptical galaxies, clusters of galaxies, bulges of spirals) **so they have a characteristic length**
- so imagine a potential of the form $-\phi(r)=-GM/\mathrm{sqrt}(r^2+b^2)$; where b is a scale length

$\nabla^2\Phi(r)=(1/r^2)\, d/dr(r^2 d\phi/dr)=3GMb^2/(r^2+b^2)^{5/2}=4\pi G\rho(r)$
[Poissons eq]

and thus

$\rho(r) =(3M/4\pi b^3)[1+(r/b)^2]^{-5/2}$ which can also be written as

- $\rho(r)=(3b^2M/4\pi)(r^2+b^2)^{-5/2}$.

Not so Simple - Plummer Potential sec 2.2 in B&T

Now take limits $r<<b$ $\rho(r) =(3GM/4\pi b^3)$ constant
$r>>b$ $\rho(r) =(3GM/4\pi b^3)r^{-5}$ finite

Plummer potential was 'first' guess at modeling 'real' spherical systems; it is one of a more general form of 'polytropes' * B&T (pg 300)

Potential energy of Plummer is $U=3\pi GM^2/32b$

*Polytropes are self-gravitating gaseous spheres that were useful as crude approximation to more realistic models-they have polytropic relation between pressure and density $P=K\rho^{(1+1/n)}$

https://www.astro.princeton.edu/~gk/A403/polytrop.pdf

Spherical systems- Plummer potential cont

- Another potential with an analytic solution is the Plummer potential - in which the density is constant near the center and drops to zero at large radii - this has been used for globular clusters, elliptical galaxies and clusters of galaxies.

- One such form- Plummer potential
  $\phi=-GM/(\mathrm{sqrt}(r^2+b^2)$; b is called a scale length

  The density law corresponding to this potential is
  (using the definition of $\nabla^2\phi$ in a spherical coordinates)

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right).$$

$\nabla^2\phi=(1/r^2)d/dr(r^2 d\phi/dr)=(3GMb^2)/((r^2+b^2)^2)^{5/2}$

$\rho(r)=(3M/4\pi b^3)(1+(r/b)^2)^{-5/2}$

Potential energy $W=-3\pi GM^2/32b$

Many More Not So Simple Analytic Forms

- There are many more forms which are better and better approximations to the true potential of 'spherical' self gravitating systems
- Another frequently used is the modified Hubble law used for clusters of galaxies

- start with a measured quantity the surface brightness distribution (more later)
  $I(r)=2aj_o(1+(r/a)^2)^{-1}$; where I(R) is the measured surface brightness

which gives a **3-D** luminosity density

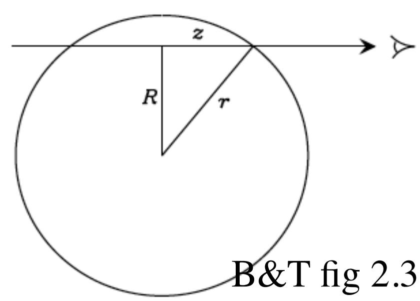
$j=j_o(1+(r/a)^2)^{-3/2}$ a is the core radius

at r=a ; $I(a)=1/2I(0)$;

- Now if light traces mass and the mass to light ratio is constant

$M=\int j(r)d^3r=$
$4\pi a^3Gj_o[\ln[R/a+\mathrm{sqrt}(1+(r/a)^2)]-(r/a)(1+(r/a)^{-1/2}]$

- and the potential is also analytic



B&T fig 2.3

Problems: mass diverges logarithimically
BUT potential is finite and at r>>a is almost GM/r

Last Spherical Potential S&G Prob 3.7

- In the last 15 years numerical simulations have shown that the density distribution of dark matter can be well described by a form called 'NFW' density distribution

  $\rho(r)=\rho(0)/[(r/a)^{\alpha}(1+(r/a))^{\beta-\alpha}]$ with $(\alpha,\beta)= (1,3)$

  Integrating to get the mass
  $M(r)=4\pi G\rho(0)a^3\ln[1+(r/a)]-(r/a)/[1+(r/a)]$
  and potential $\phi=[\ln(1+(r/a)]/(r/a)]$

  <u>See problem 3.7 in S&G</u>

The NFW density distribution is an analytic approximation to numerical simulations of cold dark matter

Other Forms

- However all the forms so far have a Keplerian rotation $v \sim r^{-1/2}$ while real galaxies have flat rotation curves $v_c(R)=v_0$

- A potential with this property must have $d\phi/dr=v_0^2/R$; $\phi=v_0^2\ln R+C$

- However this is a rather artificial form; real galaxies seem to be composed of 3 parts: disk (D), bulge (B), halo (H) and it is the <u>sum of the 3</u> that gives the flat rotation curve (very fine tuned and very flexible)

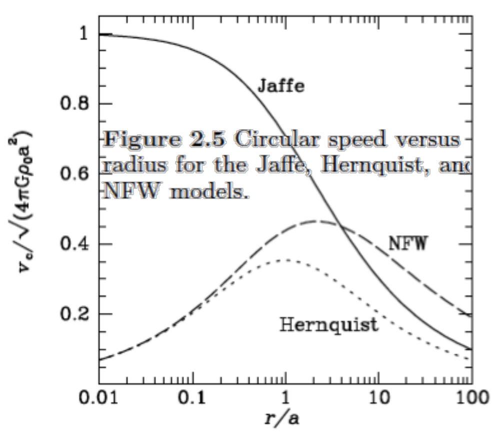
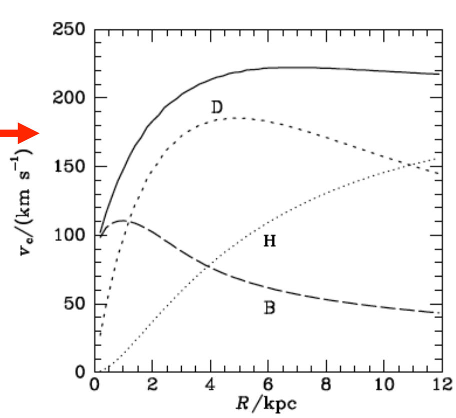


Figure 2.5 Circular speed versus radius for the Jaffe, Hernquist, and NFW models.



Summary of Dynamical Equations

- **gravitational pot'l $\Phi(r)=-G\int\rho\,(r)/|r-r'|\,d^3r$**
- **Gravitational force $F(r)= -\nabla\Phi(r)$**
- **Poissons Eq $\nabla^2\Phi(r)= 4\pi G\rho$; if there are no sources**

  Laplace Eq $\nabla^2\Phi(r)= 0$
- **Gauss's theorem** : $\int\nabla\Phi(r)\bullet ds^2=4\pi GM$
- **Potential energy $U=1/2\int r\rho(r)\nabla\Phi d^3r$**

- In words Gauss's theorem says that the integral of the normal component of $\nabla\Phi$ over and closed surface equals $4\pi G$ times the mass enclosed

Today

- Non-spherical potential
- Virial theorm
- Time scales- collisionless systems

and if time

- Collisionless Boltzmann Eq
  - Jeans equations

Potentials are Separable

- We make the fundamental assumption that the potential of a system can be decomposed into separable parts-
- This is because Poisson's equation is linear :
- differences between any two ϕ—ρ pairs is also a ϕ—ρ pair, and

differentials of ϕ—ρ or are also ϕ—ρ pairs

- e.g. $\phi_{total}=\phi_{bulge}+\phi_{disk}+\phi_{halo}$

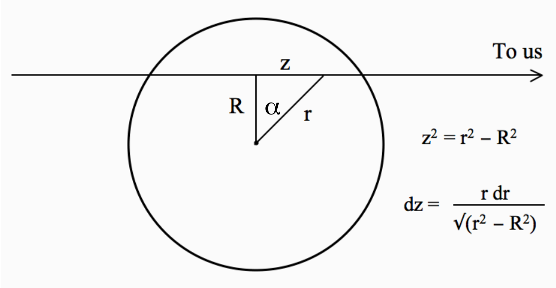
Projection Effects

from M. Whittle

http://people.virginia.edu/~dmw8f/astr5630/Topic07/Lecture_7.html

- Observed luminosity density I(R)=integral over true density distribution j(r) (in some wavelength band)
- Same sort of projection for velocity field but weighted by the density distribution of tracers
- Density distribution solution is an Abel integral (see appendix B.2 in B&T)
  - the velocity solution is also an Abel integral
- There are only a few useful I(R) & j(r) pairs that can both be expressed algebraically
  - e.g. $I(R) = I(0) / [1 + (R/r_0)^2]$ with $j(r) = I(0) / 2r_0[1 + (r/r_0)^2]^{3/2}$

$$I(R) = \int_{-\infty}^{\infty} j(r)\, dz = 2\int_R^{\infty} \frac{j(r)\, r\, dr}{\sqrt{(r^2 - R^2)}}$$



$$j(r) = \frac{-1}{\pi} \int_r^{\infty} \frac{dI}{dR} \frac{dR}{\sqrt{(R^2 - r^2)}}$$

So Far Spherical Systems

- But spiral galaxies have a significant fraction of the mass (?; at least the baryons) in a flattened system.
- Remember Oort constants

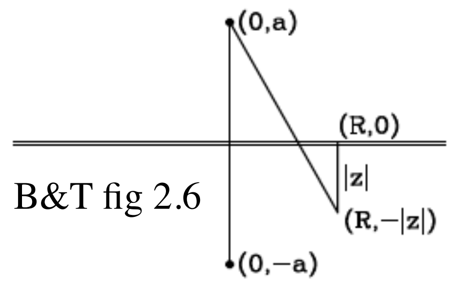
Kuzmin Disk B&T sec 2.3 S&G Prob 3.4;

- This ansatz is for a flattened system and separates out the radial and z directions
- Assume $\phi_K(z,R)$= GM/[sqrt($R^2+(a+z)^2$)] ; axisymmetric (cylindrical)

  R is in the x,y plane
- Analytically, outside the plane, ϕ_K has the form of the potential of a point mass displaced by a distance 'a' along the z axis
  – e.q. R(z)= { (0, a); z<0 ; (0, -a); z>0
- Thus $\nabla^2\Phi$=0 everywhere except along z=0- Poisson's eq
- Applying Gauss's thm $\int\nabla\Phi d^2s=4\pi GM$

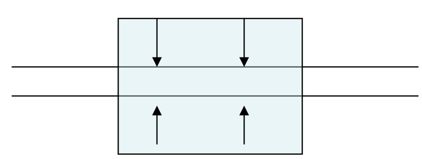
and get $\Sigma(R)=aM/[2\pi(R^2+a^2)^{3/2}]$

this is in infinitely thin disk... not too bad an approx



B&T fig 2.6

Use of Gauss's thm (divergence) the sum of all sources minus the sum of all sinks gives the net flow out of a region.



$\int\nabla\Phi d^2s=4\pi GM=2\pi G\Sigma$

as $z\rightarrow 0$; **$\Sigma=(1/2\pi)G\, d\Phi/dr$**

Isothermal Sheet MBW pg 498

- simple model for the vertical structure of disk galaxies
- Allows an estimate the disk mass from a measurement of the vertical velocity dispersion, σ_z, and the radial scale length, R_d, if one knows the vertical scale height of the tracer population
- The relevant Poisson eq is $d^2\phi_z/d\,(z/z_d)^2=1/2\exp-(\phi_z)$;
- $\phi_z=\phi/\sigma^2_z$ *and* $z_d=\sigma_z/sqrt(8\pi G\rho(R,0))$

$\sigma^2_z(R)=(z/z_d)GM_dR_d\exp(-R/R_d)$

- where z_d is the vertical scale height of the disk and R_d is the radial scale length
- can solve for the density distribution the disk

- *Why do we want to do this??- Estimates of the mass for face on galaxies where radial velocity data are impossible.*

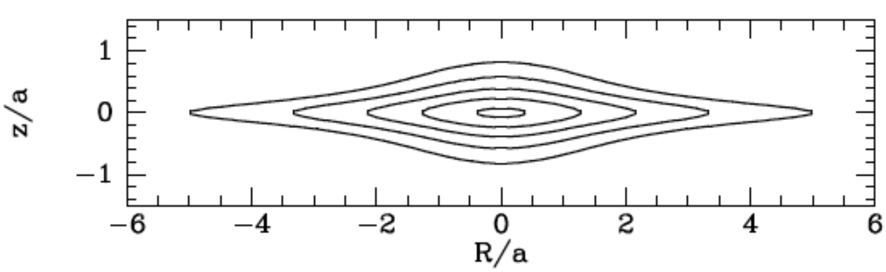
Flattened +Spherical Systems-B&T eqs

- Add the Kuzmin to the Plummer potential
- When b/a~ 0. 2, qualitatively similar to the light distributions of disk galaxies,

$$\Phi_M(R,z)=-\frac{GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2}\,)^2}}. \tag{2.69a}$$

When $a=0$, Φ_M reduces to Plummer's spherical potential (2.44a), and when $b=0$, Φ_M reduces to Kuzmin's potential of a razor-thin disk (2.68a). Thus, depending on the choice of the two parameters a and b, Φ_M can represent the potential of anything from an infinitesimally thin disk to a spherical system. If we calculate $\nabla^2\Phi_M$, we find that the mass distribution with which it is associated is (Miyamoto & Nagai 1975)

$$\rho_M(R,z)=\left(\frac{b^2M}{4\pi}\right)\frac{aR^2+(a+3\sqrt{z^2+b^2}\,)(a+\sqrt{z^2+b^2}\,)^2}{\left[R^2+(a+\sqrt{z^2+b^2}\,)^2\right]^{5/2}(z^2+b^2)^{3/2}}. \tag{2.69b}$$



Contours of equal density in the (R; z) plane for b/a=0.2

Potential of an Exponential Disk B&T sec 2.6

- As to be discussed later the light profile of the stars in most spirals has an exponential scale LENGTH

$\Sigma(R)=\Sigma_0 \exp(-R/R_d)$ (this is surface brightness NOT surface mass density)- see next page for formula's

Mass of exponential disk

$M(R)= \int\Sigma(R)Rdr = 2\pi\Sigma_0 R_d^2[1-\exp(R/R_d)(1+R/R_d)]$

when R gets large $M\sim 2\pi\Sigma_0 R_d^2$

Potential of an Exponential Disk B&T sec 2.6

.

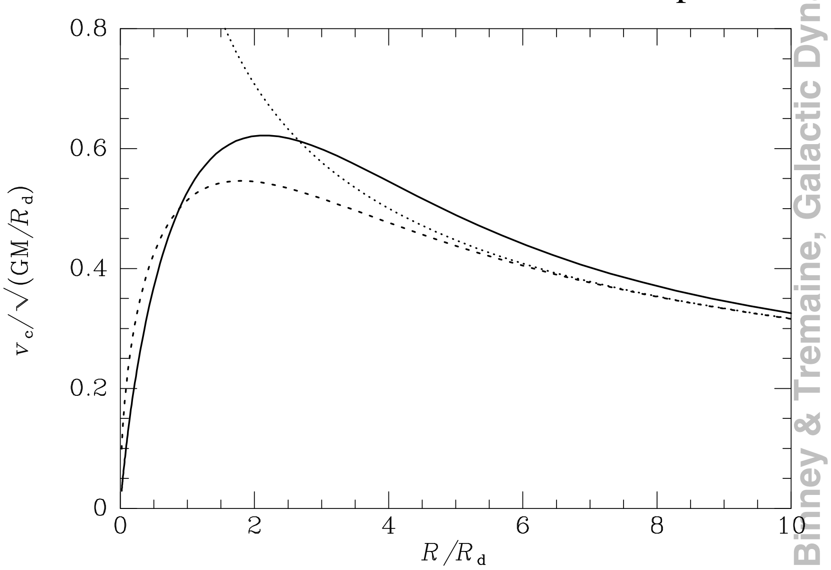
The circular velocity peaks at $R\sim 2.16\ R_d$ - approaches Keplerian for a point mass at large R (eq. 11.30 in MWB) and depends only on Σ_0 and R_d

As long as the vertical scale length is much less than the radial scale the vertical distribution has a small effect - e.g. separable effects !

IF the disk is made only of stars (no DM) and and if they all have the same mass to light ratio Γ , R_d is the scale length of the stars , then the observables I_0,R_d,$v_{circ}(r)$ have all the info to calculate the mass!

Circular Velocity for 3 Potentials

Exponential disk (Solid line)
Point mass (.....)
Spherical exponential (-----)



Explaining Disks

- **Most important properties of disk dominated galaxies**
  - **Brighter disks are on average**
    - **larger, redder, rotate faster, smaller gas fraction**
  - **flat rotation curves**
  - **surface brightness profiles close to exponential**
  - **lower metallicity in outer regions**

For a uniform disk (S&G Prob 3.24) the potential is

− $\phi(x) = 2\pi G\Sigma|z|$. Σ is the surface density z=height above disk

Explaining Disks

- Traditional to model disk as an infinitely thin exponential disk with a surface density distribution $\Sigma(R)=\Sigma_{0y}\exp(-R/R_d)$
  - This gives a potential (MBW pg 496) which is a bit messy

$$\phi(R, z)=-2\pi G\Sigma_0{}^2R_D\int[J_0(kR)\exp(-k|z|)]/[1+(kR_D)^2]^{3/2}dk$$

(do not need to know this...)

Modeling Spirals

- As indicated earlier to fit the observed density and velocity distributions in the MW one needs a 3 component mass distribution
- Traditionally this is parameterized as the sum of
  - disk $\Sigma(R) =\Sigma_0[\exp-R/a]$
  - spheroid (bulge) using $I(R)=I_0R_s{}^2/[R+Rs]^2$ or similar forms
  - dark matter halo $\rho(r)=\rho(0)/[1+(r/a)^2]$
- See B&T sec 2.7 for more complex forms- 2 solutions in B&T- notice extreme difference in importance of halo (H) (table 2.3)