Dynamics and how to use the orbits of stars to do interesting things

chapter 3 of S+G- parts of ch 2 of B&T and parts of Ch 11 of MWB

Galactic Rotation- Oort Constants

- using a bit of trig

$R(\cos\alpha)= R_0\sin(l)$

$R(\sin\alpha)= R_0\cos(l)-d$

so

$V_{observered,radial}=(\omega- \omega_0)\, R_0\sin(l)$

$V_{observered,tang}=(\omega- \omega_0)\, R_0\cos(l)-\omega d$

then following the text expand $(\omega- \omega_0)$ around R_0 and using the fact that most of the velocities are local e.g. *R-R_0 is small and d is smaller than R or R_0* (not TRUE for HI) and some more trig

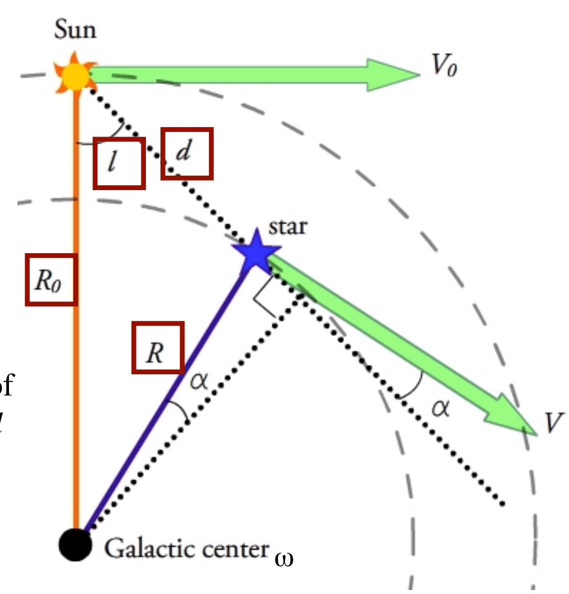
get

$V_{observered,radial}=Ad\sin(2l); V_{obs,tang}=Ad\cos(2l)+Bd$

Where

$A=-1/2\ R_0\ (d\omega/dr)$ at R_0

$B=-1/2\ R_0\ (d\omega/dr - \omega)$



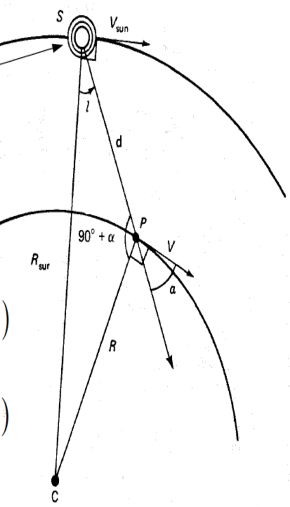
Galactic Rotation Curve- sec 2.3.1 S+G

Assume gas/star has a perfectly circular orbit

At a radius R_0 orbit with velocity V_0; another star/ parcel of gas at radius R has a orbital speed V(R)

since the angular speed V/R drops with radius, V(R) is positive for nearby objects with galactic longitude l 0<l<90 etc etc (pg 91 bottom)

- Galactic Rotation Curve
  - At R_{sun} the lsr has a velocity of V_0
  - A star at P has an apparent velocity of



1) $V_r = V\cos(\alpha) - V_0\sin(l)$

2) $V_r = R_0\sin(l)\left(\frac{V}{R} - \frac{V_0}{R_0}\right)$

- Convert to angular velocity ω
- $V_{observered,radial} = \omega R(\cos\alpha) - \omega_0 R_0 \sin(l)$
- $V_{observered,tang} = \omega R(\sin\alpha) - \omega_0 R_0 \cos(l)$

In terms of Angular Velocity

- model Galactic motion as circular motion with monotonically decreasing angular rate with distance from center.
- Simplest physics: if the mass of the Galaxy is all at center angular velocity ω at R is $\omega = M^{1/2}G^{1/2}R^{-3/2}$
- If looking through the Galaxy at an angle l from the center, velocity at radius R projected along the line of site minus the velocity of the sun projected on the same line is

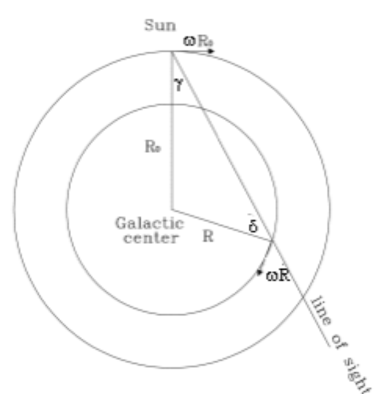
  (1) $V = \omega R \sin d - \omega_o R_o \sin l$

  ω = angular velocity at distance R

  ω_o = angular velocity at a distance R_o

  R_o = distance to the Galactic center

  l = Galactic longitude



- Using trigonometric identity $\sin d = R_o \sin(l/R)$

  and substituting into equation (1)

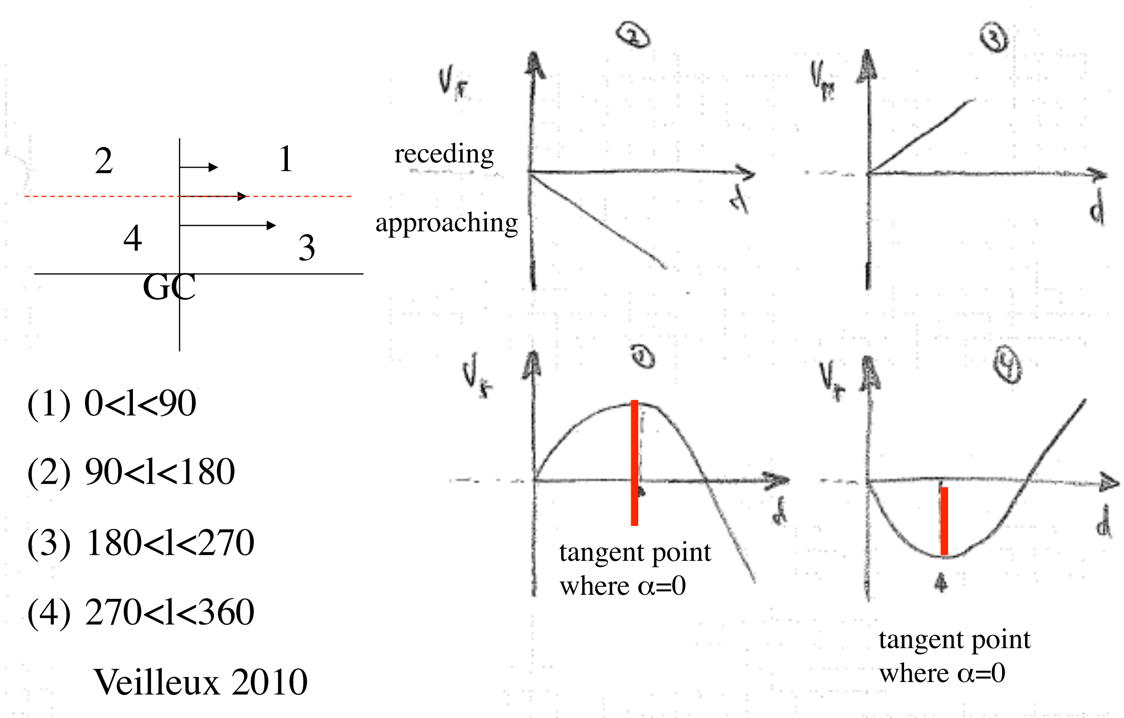
- $V = (\omega - w\,\omega_o)\, R_o \sin l$

http://www.haystack.mit.edu/edu/undergrad/srt/SRT Projects/rotation.html

Continued

- The tangential velocity $v_T = V_o \sin\alpha - V_o \cos l$
  and $R \sin\alpha = R_o \cos l - d$
- a little algebra then gives
  $V_T = V/R(R_o \cos l - d) - V_o \cos l$
- re-writing this in terms of angular velocity
  $V_T = (\omega - \omega_o) R_o \cos l - \omega d$

- For a reasonable galactic mass distribution we expect that the angular speed $\omega = V/R$ is monotonically decreasing at large R (most galaxies have flat rotation curves (const V) at large R) then get a set of radial velocities as a function of where you are in the galaxy
- V_T is positive for $0<l<90$ and nearby objects- if $R>R_0$ it is negative
- For $90<l<180$ V_T is always negative
- For $180<l<270$ V_T is always positive (S+G sec 2.3.1)



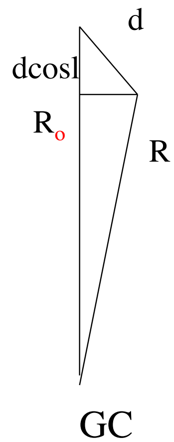
Oort Constants S&G pg 92-93

Derivation:

- for objects near to sun, use a Taylor series expansion of $\omega-\omega_o$

$\omega-\omega_o = d\omega/dR\,(R-R_o)$

$\omega = V/R$; $d\omega/dR = d/dr(V/R) = (1/R)dV/dr - V/R^2$



then to first order

$V_r = (\omega-\omega_o)R_o \sin l = [dV/dr - V/R](R-R_o) \sin l$; when $d \ll R_o$

$R-R_o = d\cos l$ which gives

$V_r = (V_o/R_o - dV/dr)d \sin l \cos l$

using trig identity $\sin l \cos l = 1/2 \sin 2l$

one gets the Oort forumla

$V_r = Ad\sin 2l$ where $A = \frac{1}{2}\left[\frac{V_o}{R_o} - \left(\frac{dV}{dR}\right)_{R_o}\right]$

One can do the same sort of thing for V_T

Oort Constants

- For nearby objects ($d \ll R$)
  - $V(R) \sim R_0 \sin l\,(d(V/R)/dr)(R-R_0)$

  $\sim d\sin(2l)[-R/2(d(V/R)/dr) \sim d\mathbf{A}\sin(2l)$

(l is the galactic longitude)

- A is one of 'Oorts constants'
- The other B (pg 93 S+G) is related to the tangential velocity of a object near the sun $V_t = d[A\cos(2l) + B]$
- So, stars at the same distance r will show a systematic pattern in the magnitude of their radial velocities across the sky with Galactic longitude.
- A is the Oort constant describing the shearing motion and B describes the rotation of the Galaxy

$$A = \frac{1}{2}\left[\frac{V_o}{R_o} - \left(\frac{dV}{dR}\right)_{R_o}\right]$$

$$B = -\frac{1}{2}\left[\frac{V_o}{R_o} + \left(\frac{dV}{dR}\right)_{R_o}\right]$$

$$A + B = -\left(\frac{dV}{dR}\right)_{R_o} \quad ; \quad A - B = \frac{V_o}{R_o}$$

$A = -1/2[R\,d\omega/dr]$

Useful since if know A get kinematic estimate of d

Radial velocity $v_r \sim 2AR_0(1-\sin l)$ only valid near $l \sim 90$ measure $AR_0 \sim 115$km/s

Oort 'B'

- B measures 'vorticity' $B=-(\omega=-1/2[Rd\omega/dr])=-1/2[(V/R)+(dV/dR)]$ angular momentum gradient

  $\omega=A-B=V/R$; angular speed of Local standard of rest (sun's motion)

  Oort constants are local description of differential rotation

  Values

  A=14.8 km/s/kpc

  B=-12.4 km/s/kpc

  Velocity of sun $V_0=R_0(A-B)$

  I will not cover epicycles (stars not on perfect circular orbits) now (maybe next lecture): : see sec pg 133ff in S&G

A Guide to the Next Few Lectures

•The geometry of gravitational potentials : methods to derive gravitational potentials from mass distributions, and visa versa.

•Potentials define how stars move
consider stellar orbit shapes, and divide them into orbit classes.

•The gravitational field and stellar motion are interconnected :
the Virial Theorem relates the global potential energy and kinetic energy of the system.

- The Distribution Function (DF) :
  the DF specifies how stars are distributed throughout the system and with what velocities.

For collisionless systems, the DF is constrained by a continuity equation :
the Collisionless Boltzmann Equation

•This can be recast in more observational terms as the Jeans Equation.
The Jeans Theorem helps us choose DFs which are solutions to the continuity equations

*Adapted from M. Whittle

A Reminder of Newtonian Physics sec 2.1 in B&T

Newtons law of gravity tells us that two masses attract each other with a force

$$\frac{d}{dt}(m\boldsymbol{v}) = -\frac{GmM}{r^3}\boldsymbol{r}$$

ϕ (x) is the potential

If we have a collection of masses acting on a mass m_α the force is

$$\frac{d}{dt}(m_\alpha \boldsymbol{v}_\alpha) = -\sum_\beta \frac{G m_\alpha M_\beta}{|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta|^3}(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta), \ \alpha \neq \beta$$

$$\frac{d}{dt}(m\boldsymbol{v}) = -m\nabla\Phi(\boldsymbol{x}),$$

with

$$\Phi(\boldsymbol{x}) = -\sum_\alpha \frac{G m_\alpha}{|\boldsymbol{x} - \boldsymbol{x}_\alpha|}, \ for\ \boldsymbol{x} \neq \boldsymbol{x}_\alpha$$

Gauss's thm $\int \nabla\phi \bullet ds^2 == 4\pi G M$
the Integral of the normal component over a closed surface $= 4\pi G$ x mass within that surface

the gravitational potential. If we can approximate the discrete stellar distribution with a continuous distribution ρ.

$$\Phi(\boldsymbol{x}) = -\int \frac{G\rho(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3\boldsymbol{x}'$$

Conservation of Energy and Angular Momentum Sec 3.1 S&G

In the absence of external forces a star will conserve energy along its orbit

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v}) = -m\,\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}),$$

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v}) + m\,\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}) = 0$$

But since $\frac{d\Phi}{dt} = \boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x})$

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

where $\{\hat{x}, \hat{y}, \hat{z}\}$ are the unit vectors in their respective directions.

$$\frac{d}{dt}\left[\frac{m}{2}(\boldsymbol{v}^2) + m\Phi(\boldsymbol{x})\right] = 0$$

This is just the KE + PE

$$\frac{dL}{dt} = \boldsymbol{x}\times m\frac{d\boldsymbol{v}}{dt} = -m\,\boldsymbol{x}\times\nabla\Phi$$

Angular momentum L

Some Basics - M. Whittle

- The gravitational potential energy is a scalar field
- its gradient gives the net gravitational force (per unit mass) which is a vector field :

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla = \nabla^2$$

$$\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} \, d^3\mathbf{r}'$$

$$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = G \int_V \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} \, \rho(\mathbf{r}') \, d^3\mathbf{r}'$$

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = -4\pi G\rho(\mathbf{r})$$

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})$$ ⟷ Poissons eq inside the mass distribution

$$\nabla^2\Phi(\mathbf{r}) = 0$$ ⟷ Outside the mass dist

Poisson's Eq+ Definition of Potential Energy (W)

So the force per unit mass is

ρ(x) is the density dist

$$\boldsymbol{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = \int G\rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}|^3} d^3\mathbf{x}'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides

S+G pg 112-113

$$\nabla^2\Phi(\mathbf{x}) = -\nabla^2 \int \frac{G\rho(\mathbf{x}')}{|\mathbf{x}-x'|} d^3\mathbf{x}'$$

$$= 4\pi G\rho(\mathbf{x})$$ Poisson's equation.

Potential energy W

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\,\Phi(\mathbf{r})\,d^3\mathbf{r} = -\frac{1}{8\pi G}\int_V |\nabla\Phi|^2\,d^3\mathbf{r}$$

Derivation of Poisson's Eq

So the force per unit mass is

$$F(x)=-\nabla\Phi(x)=\int G\rho(x')\frac{(x-x')}{|x-x|^3}d^3x'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides

$$\nabla^2\Phi(x)=-\nabla^2\int\frac{G\rho(x')}{|x-x'|}d^3x'$$

$$=4\pi G\rho(x)$$ **Poisson's equation.**

see S+G pg112 for detailed derivation

Characteristic Velocities

$v^2_{circular}$=r dΦ(r)/dt=GM/r; v=sqrt(GM/r) Keplerian

velocity dispersion **$\sigma^2=(1/\rho)\int\rho\,(\partial\Phi(r,z)/\partial z)dz$**
or alternatively $\sigma^2(R)=(4\pi G/3M(R)\int r\rho(r)\,M(R)\,dr$

escape speed $=v_{esc}$=sqrt(2Φ(r)) or $\Phi(r)=1/2v^2_{esc}$
so choosing r is crucial

More Newton-Spherical Systems B&T 2.2

Newtons 1st theorem: a body inside a spherical shell has no net gravitational force from that shell; e.g. $\nabla\Phi(r)=0$

Newtons 2nd theorem: the gravitational force on a body outside a spherical shell is the same as if all the mass were at a point at the center of the shell.

Simple examples:
Point source of mass M; potential $\Phi(r) = -GM/r$;
definition of circular speed; speed of a test particle on a circular orbit at radius r
$\mathbf{v^2_{circular}=r\, d\Phi(r)/dt}=GM/r$; $v_{circular}=\mathrm{sqrt}(GM/r)$;Keplerian

velocity dispersion $\boldsymbol{\sigma^2=(1/\rho) \int\rho\, (\partial\Phi(r,z)/\partial z)dz}$
escape speed $=\mathrm{sqrt}[2\Phi(r)]=\mathrm{sqrt}(2GM/r)$; from equating kinetic energy to potential energy $1/2mv^2=|\Phi(r)|$

Escape Speed

- As r goes to infinity $\phi(r)$ goes to zero
- so to escape $v^2>2\phi(r)$; e.q. $v_{esc}=\mathrm{sqrt}(-2\phi(r))$

Gravity and Dynamics-Spherical Systems- Repeat

- Newtons 1st theorm : a body inside a a spherical shell has no net force from that shell $\nabla\phi$ =0
- Newtons 2nd theorm ; a body outside the shell experiences forces as if they all came from a point at the center of the shell-Gravitational force at a point outside a closed **sphere** is the same as if all the mass were at the center
    - This does not work for a thin disk- cannot ignore what is outside of a given radius
- One of the prime observables (especially for spirals) is the circular velocity; in general $V^2(R)/R=G(M<R)/R^2$

  more accurate estimates need to know shape of potential
- so one can derive the mass of a flattened system from the rotation curve

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- point source has a potential ϕ=-GM/r
- A body in orbit around this point mass has a circular speed v_c^2=r ϕd/dr=GM/r
- v_c=sqrt(GM/r); Keplerian
- Escape speed from this potential v_{escape}=sqrt(2ϕ)=sqrt(2GM/r) (conservation of energy KE=1/2mv^2_{escape}

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Homogenous Sphere B&T sec 2.2.2

- Constant density sphere of radius a and density ρ_0
- M(r)=4πG$r^3\rho_0$; r<a
- M(r)=4πG$a^3\rho_0$; r>a

  ϕ(R)=-d/dr(M(R)) ; ϕ(R)=-3/5GM^2/R ; B&T 2.41)

  R>a ϕ(r)=4πG$a^3\rho_0$=-GM/r

  R<a ϕ(r)=-2πGρ_0(a^2-1/3r^2));

v^2_{circ}= (4π/3)Gρ_0r^2 ; <u>solid body rotation</u> R<a

Orbital period **T=2πr/v_{circ}=sqrt(3π/Gρ_0)**

Dynamical time=crossing time =T/4=**sqrt(3π/16Gρ_0)**

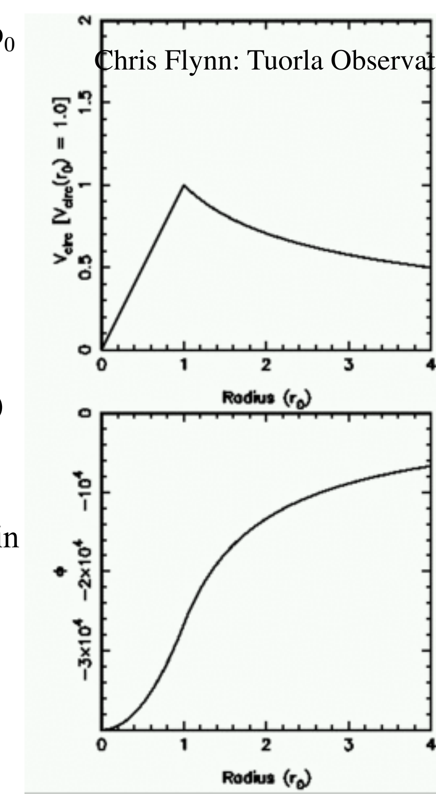
Potential is the same form as an harmonic oscillator with angular freq 2π/T (B&T 2.2.2(b))

Regardless of r a particle will reach r=0 (in free fall) in a time T=/4

Eq of motion of a test particle INSIDE the sphere is

dr^2/dt^2=-GM(r)/r^2=-(4π/3)Gρ_0r

General result dynamical time ~sqrt(1/Gρ)



Some Simple Cases

- **Constant density sphere** of radius a and density ρ_0 continued

Potential energy (B&T) eq 2.41, 2.32

$\phi(R)$=-d/dr(M(R)) ;

R>a $\phi(r)=4\pi Ga^3\rho_0=-GM/r$

R<a $\phi(r)=-2\pi G\rho_0(a^2-1/3r^2)$);

$v^2_{circ}=(4\pi/3)G\rho_0 r^2$ solid body rotation

Potential is the same form as a harmonic oscillator

e.g. the eq of motion is $d^2r/dt^2=-GM(r)/r=4\pi/3Gr\rho$; solution to harmonic oscillator is

$r=A\cos(\omega t+\phi)$ with ω= sqrt($4\pi/3G\rho$)=$2\pi/T$

T=sqrt($3\pi/G\rho_0$)=$2\pi r/v_{circ}$

Spherical Systems: Homogenous sphere of radius 'a' Summary

- $M(r)=4/3\pi r^3\rho$ (r<a); r>a $M(r)=4/3\pi r^3 a$
- Inside body (r<a); $\phi(r)=-2\pi G\rho(a^2-1/3\ r^2)$ (from eq. 2.38 in B&T)

Outside (r>a);)$\phi(r)=-4\pi G\rho(a^3/3)$

Solid body rotation $v_c^2=-4\pi G\rho(r^2/3)$

Orbital period $T=2\pi r/v_c$=sqrt($3\pi/G\rho$);

a crossing time (dynamical time) =T/4=sqrt($3\pi/16G\rho$)

potential energy $W=-3/5GM^2/a$

The motion of a test particle inside this sphere is that of a simple harmonic oscillator $d^2r/dt^2=-G(M(r)/r^2=4\pi G\rho r/3$ with angular freq $2\pi/T$

no matter the intial value of r, a particle will reach r=0 in the dynamical time T/4

In general the dynamical time $t_{dyn}\sim1/\text{sqrt}(G\langle\rho\rangle)$

and its 'gravitational radius' $r_g=GM^2/W$

Summary of Dynamical Equations

- **gravitational pot'l $\Phi(r)=-G\int\rho\,(r)/|r-r'|\,d^3r$**
- **Gravitational force $F(r)= -\nabla\Phi(r)$**
- **Poissons Eq $\nabla^2\Phi(r)= 4\pi G\rho$; if there are no sources**

  Laplace Eq $\nabla^2\Phi(r)= 0$
- **Gauss's theorem** : **$\int\nabla\Phi(r)\bullet ds^2=4\pi GM$**
- **Potential energy $W=1/2\int r\rho(r)\nabla\Phi d^3r$**

- In words Gauss's theorem says that the integral of the normal component of $\nabla\Phi$ over a closed surface equals $4\pi G$ times the mass enclosed

Potentials are Separable

- We make the fundamental assumption that the potential of a system can be decomposed into separable parts-
- This is because Poisson's equation is linear :
- differences between any two $\phi-\rho$ pairs is also a $\phi-\rho$ pair, and

differentials of $\phi-\rho$ or are also $\phi-\rho$ pairs

- e.g. $\phi_{total}=\phi_{bulge}+\phi_{disk}+\phi_{halo}$

So Far Spherical Systems

- But spiral galaxies have a significant fraction of the mass (?; at least the baryons) in a flattened system.

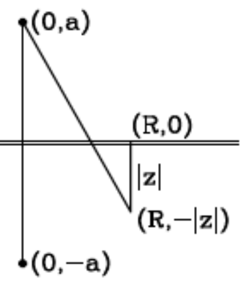
Kuzmin Disk B&T sec 2.3 S&G Prob 3.4;

- This ansatz is for a flattened system and separates out the radial and z directions
- Assume $\phi_K(z,R)$= GM/[sqrt($R^2+(a+z)^2$)] ; axisymmetric (cylindrical)

  R is in the x,y plane
- Analytically, outside the plane, ϕ_K has the form of the potential of a point mass displaced by a distance 'a' along the z axis
  - e.q. R(z)= $\begin{cases} (0, a); z<0 \\ (0, -a); z>0 \end{cases}$
- Thus $\nabla^2\Phi=0$ everywhere except along z=0- Poisson's eq
- Applying Gauss's thm $\int\nabla\Phi d^2s=4\pi GM$

and get $\Sigma(R)=aM/[2\pi(R^2+a^2)^{3/2}]$

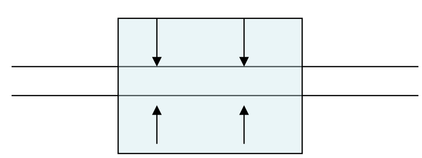
this is in infinitely thin disk... not too bad an approx



B&T fig 2.6

Use of Gauss's thm (divergence)

the sum of all sources minus the sum of all sinks gives the net flow out of a region.



$\int\nabla\Phi d^2s=4\pi GM=2\pi G\Sigma$

as z $\longrightarrow$ 0 ; **$\Sigma=(1/2\pi)G\, d\Phi/dr$**

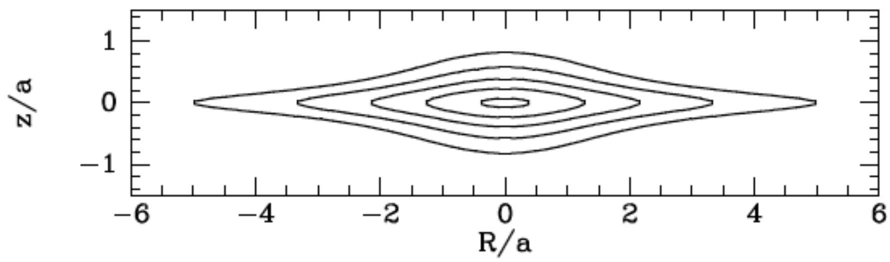
Flattened +Spherical Systems-Binney and Tremaine eqs

- Add the Kuzmin to the Plummer potential (S&G 113,114)
- When b/a~ 0. 2, qualitatively similar to the light distributions of disk galaxies,

$$\Phi_M(R,z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2+b^2})^2}}. \qquad (2.69a)$$

When $a = 0$, Φ_M reduces to Plummer's spherical potential (2.44a), and when $b = 0$, Φ_M reduces to Kuzmin's potential of a razor-thin disk (2.68a). Thus, depending on the choice of the two parameters a and b, Φ_M can represent the potential of anything from an infinitesimally thin disk to a spherical system. If we calculate $\nabla^2\Phi_M$, we find that the mass distribution with which it is associated is (Miyamoto & Nagai 1975)

$$\rho_M(R,z) = \left(\frac{b^2M}{4\pi}\right)\frac{aR^2 + (a + 3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{\left[R^2 + (a+\sqrt{z^2+b^2})^2\right]^{5/2}(z^2+b^2)^{3/2}}. \qquad (2.69b)$$



Contours of equal density in the (R; z) plane for b/a=0.2

Explaining Disks

- Remember the most important properties of disk dominated galaxies (MBW pg 495)
  – More luminous disks are on average
    - larger, redder, rotate faster, smaller gas fraction
  – flat rotation curves
  – surface brightness profiles close to exponential
  – lower metallicity in outer regions
  – traditional to model them as an infinitely thin exponential disk with a surface density distribution $\Sigma(R)=\Sigma_{0y}\exp(-R/\mathbf{R_d})$

– This gives a potential (MBW pg 496) which is a bit messy

$\phi(R, z)=-2\pi G\Sigma_0{}^2 R_D\int[J_0(kR)\exp(-k|z|)]/[1+(k\mathbf{R_D})^2]^{3/2}dk$

J_0 is a Bessel function order zero

Modeling Spirals

- to fit the observed density and velocity distributions in the MW one needs a 3 component mass distribution
- Traditionally this is parameterized as the sum of
  - disk $\Sigma(R) = \Sigma_0[\exp{-R/\mathbf{R_d}}]$
  - spheroid (bulge) using $I(R)=I_0 R_s^2/[R+R_s]^2$ or similar forms
  - dark matter halo

    $\rho(r)=\rho(0)/[1+(r/a)^2]$
- See B&T sec 2.7 for more complex forms- 2 solutions in B&T- notice extreme difference in importance of halo (H) (table 2.3)