Collisionless Boltzmann Eq (Vlasov eq)
S+G sec 3.4

- When considering the structure of galaxies cannot follow each individual star (10^{11} of them!),
- Consider instead stellar density and velocity distributions. However, a fluid model is not really appropriate since a fluid element has a single velocity, which is maintained by particle-particle collisions on a scale much smaller than the element. For stars in the galaxy this **not true**-stellar collisions are very rare, and even encounters where the gravitational field of an individual star is important in determining the motion of another are very infrequent
- So taking this to its limit, treat each particle as being collisionless, moving under the influence of the mean potential generated by **all** the other particles in the system $\phi(x,t)$
  - (e.g. in galaxies we use stars as 'test particles' since it seems that most of the potential is produced by dark matter)

Collisionless Boltzmann Eq S&G 3.4

- The distribution function is defined such that $f(\mathbf{r},\mathbf{v},t)d^3\mathbf{x}d^3\mathbf{v}$ specifies the number of stars inside the volume of phase space $d^3\mathbf{x}d^3\mathbf{v}$ centered on $(\mathbf{x},\mathbf{v})$ at time t-

At time t a full description of the state of this system is given by specifying the number of stars

$f(x, v, t)d^3xd^3v$

Then $f(\mathbf{x}, \mathbf{v}, t)$ is called the "distribution function" (or "phase space number density") in 6 dimensions (**x** and **v**) of the system.

$f \geq 0$ since no negative star densities

Since potential is smooth nearby particles in phase space move together-- **fluid approx.**

For a collisionless stellar system in dynamic equilibrium, the gravitational potential, ϕ , relates to the phase-space distribution of stellar tracers $f(\mathbf{x}, \mathbf{v}, t)$, via the collisionless Boltzmann Equation

number of particles

- $n(\mathbf{x},t)=\int f(x, v, t)d^3\mathbf{v}$

- average velocity

$\langle v(\mathbf{x},t)\rangle=\int f(x, v, t)\, \mathbf{v}d^3\mathbf{v}/f(x, v, t)d^3\mathbf{v}=$
$(1/n(\mathbf{x},t))\int f(x, v, t)\, \mathbf{v}d^3\mathbf{v}$

See S&G sec 3.4

- The collisionless Boltzmann equation is like the equation of continuity,

$\partial n/\partial t + \partial(nv)\partial x = 0$: but it allows for changes in velocity and relates the changes in f (x, v, t) to the forces acting on individual stars

In one dimension CBE is

- $\boldsymbol{\partial f/\partial t + v\, \partial f/\partial x - \partial\phi(x, t) /\partial x(x, t)\, \partial f/\partial v = 0}$ eq 3.86

- For derivation of CBE see S&G pgs 142-143

Collisionless Boltzmann Eq

- This results in (S+G pg 143) eq 3.87

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0,}$$

- the flow of stellar phase points through phase space is incompressible – the phase-space density of points around a given star is always the same

- The distribution function f is a function of seven variables, so solving the collisionless Boltzmann equation in general is hard. So need either simplifying assumptions (usually symmetry), or try to get insights by taking moments of the equation.
- Moment of an eq-- multipying f by powers of v

define n(x,t) as the number density of stars at position x

then the first moment is
$\partial n/\partial t+\partial/\partial x(nv)=0$; the same eq as continuity equation of a fluid

second moment
$n\partial v/dt+nv\partial v/dx=-n\partial\phi/\partial x-\partial/\partial x(n\sigma^2)$
σ is the velocity dispersion (eq. 3.90)

Analogous to eqs of fluid mechanics but unlike fluids do not have thermodynamics to help out.... nice math but not clear how useful

Analogy with Gas- continuity eq see MBW sec 4.1.4

- $\partial\rho/\partial t + \nabla\bullet(\rho v) = 0$ which is equiv to

  $\partial\rho/\partial t + v\nabla\bullet\rho = 0$
- In the absence of encounters f satisfies the continuity eq , flow is smooth, stars do no jump discontinuously in phase space

- Continuity equation :

define w=(x,v) pair (generalize to 3-D)

dw/dt=($v,\nabla\phi$)

- $\partial f/\partial t + \nabla_6(f\ dw/dt) = 0$

- the phase-space density $f(\mathbf{x}, \mathbf{v}, t)$ around any particular star remains constant along its orbit

Analogy of Stellar Systems to Gases

- Discussion due to Mark Whittle

- similarities :

comprise many, interacting objects which act as points (separation >> size)

can be described by distributions in space and velocity eg Maxwellian velocity distributions; uniform density; spherically concentrated etc.

Stars or atoms are neither created nor destroyed -- they both obey continuity equations- [not really true, galaxies are growing systems!]

All interactions, as well as the system as a whole obeys conservation laws (eg energy, momentum)**if isolated**

- But :
- The relative importance of short and long range forces is radically different :
  - atoms interact only with their neighbors, however
  - stars interact continuously with the entire ensemble via the long range attractive force of gravity
- eg for a uniform medium : **~equal force from all distances**

Analogy of Stellar Systems to Gases
- Discussion due to Mark Whittle

- The relative frequency of strong encounters is radically different :
- -- for atoms, encounters are frequent and all are strong (ie $\delta V \sim V$)
- -- for stars, pairwise encounters are very rare, and the stars move in the smooth global potential (e.g. S+G 3.2)

- Some parallels between gas (fluid) dynamics and stellar dynamics many of the same equations **can** be used :
- ---> concepts such as Temperature and Pressure can be applied to stellar systems
- ---> we use analogs to the equations of fluid dynamics and hydrostatics
- there are also some interesting differences
- ---> pressures in stellar systems can be anisotropic
- ---> stellar systems have negative specific heat and left alone can evolve away from uniform temperature.

Collisionless Boltzmann Eq

astronomical structural and kinematic observations provide information only about the **projections of phase space distributions along lines of sight**, limiting knowledge about f and hence also about ϕ.

Therefore all efforts to translate existing data sets into constraints f and ϕ involve simplifying assumptions.
• dynamic equilibrium,
• symmetry assumptions
• particular functional forms for the distribution function and/or the gravitational potential.

Jeans Equations MBW sec 5.4.3

- Since f is a function of 7 variables obtaining a solution is challenging
- Take moments (e.g. integrate over all **v**)
- let n be the space density of 'stars'

$\partial n/\partial t+\partial(n\langle v_i\rangle)/\partial x_i=0$; continuity eq. zeroth moment

first moment (multiply by v and integrate over all velocities)

$\partial(n\langle v_j\rangle/\partial t)+\partial(n\langle v_i v_j\rangle)/\partial x_i+n\partial\phi/\partial x_j=0$

equivalently

$n\partial(\langle v_j\rangle/\partial t)+n\langle v_i\rangle\,\partial\langle v_j\rangle/\partial x_i=-n\partial\phi/\partial x_j-\partial(n\sigma^2_{ij})/\partial x_i$

n is the integral over velocity of f; $n=\int f\, d^3v$

$\langle v_i\rangle$ is the mean velocity in the i^{th} direction $=1/n\int f\, v_i\, d^3v$

the term $-\partial(n\sigma^2_{ij})/\partial x_i$ is like a pressure, but allows for different pressures in different directions- important in elliptical galaxies and bulges 'pressure supported' systems

Jeans Equations Another Formulation

- Jeans equations follow from the collisionless Boltzmann equation; Binney & Tremaine (1987), MBW 5.4.2. S+G sec 3.4 .

cylindrical coordinates and assuming an axi-symmetric and steady-state system, the accelerations in the radial (R) and vertical (Z) directions can be expressed in terms of observable quantities:

the stellar number density distribution ν_*

and 4 velocity components

a rotational velocity v_ϕ

and 4 components of random velocities (velocity dispersion components)

$\sigma_{\phi\phi}, \sigma_{RR}, \sigma_{ZZ}, \sigma_{RZ}$

$$a_R=\sigma^2_{RR}\times\frac{\partial(\ln\nu)}{\partial R}+\frac{\partial\sigma^2_{RR}}{\partial R}+\sigma^2_{RZ}\times\frac{\partial(\ln\nu)}{\partial Z}+\frac{\partial\sigma^2_{RZ}}{\partial Z}+\frac{\sigma^2_{RR}}{R}-\frac{\sigma^2_{\phi\phi}}{R}-\frac{\overline{v_\phi}^2}{R},$$

$$a_Z=\sigma^2_{RZ}\times\frac{\partial(\ln\nu)}{\partial R}+\frac{\partial\sigma^2_{RZ}}{\partial R}+\sigma^2_{ZZ}\times\frac{\partial(\ln\nu)}{\partial Z}+\frac{\partial\sigma^2_{ZZ}}{\partial Z}+\frac{\sigma^2_{RZ}}{R}.$$

where a_Z, a_R are accelerations in the appropriate directions-

given these values (which are the gradient of the gravitational potential), *the dark matter contribution can be estimated after accounting for the contribution from visible matter*

Spherical systems- Elliptical Galaxies and Globular Clusters

- For a spherical system the Jeans equations simplify to

$(1/n)d/dr\ (n\langle v^2_r\rangle) + 2\beta\langle v^2_r\rangle/r = -GM(R)/r^2$

- where $G(M(R)/r^2$ is the potential and $n(r)$, $\langle v^2_r\rangle$ and $\beta(r)$ describe the 3-dimensional density, radial velocity dispersion and orbital anisotropy of the tracer component (stars)

  $\beta(r) = 1-\langle v^2_\theta\rangle/\langle v^2_r\rangle$; $\beta = 0$ is isotropic, $\beta=1$ is radial
- We can then present the mass profile as
- $GM(r) = r\langle v^2_r\rangle\ (d\ln n/d\ln r + d\ln\langle v^2_r\rangle/d\ln r + 2\beta)$
- while apparently simple we have 3 sets of unknowns $\langle v^2_r\rangle$, $\beta(r)$, $n(r)$
- and 2 sets of observables I(r)- surface brightness of radiation (in some wavelength band) and the lines of sight projected velocity field (first moment is velocity dispersion)
- It turns out that one has to 'forward fit'- e.q. propose a particular form for the unknowns and fit for them.

This will become very important for elliptical galaxies

Motion Perpendicular to the Plane- Alternate Analysis-(S+G pgs140-145, MBW pg 163)

For the motion of stars in the vertical direction only-stars whose motions carry them out of the equatorial plane of the system.

$d/dz[n_*(z)\sigma_z(z)^2] = -n_*(z)d\phi(z,R)/dz$; where **$\phi(z,R)$ is the vertical grav potential**

The study of such general orbits in axisymmetric galaxies can be reduced to a two-dimensional problem by exploiting the conservation of the z-component of angular momentum of any star

S&G EQ 3.91 $\quad d/dz[n(z)\sigma_Z^2] = d\phi(z)/dz\ n(z)$

Use Poisson's eq $\nabla^2\Phi(r) = 4\pi G\rho$ and assume axisymmetry: ρ and ϕ depend on on (z,R)

gives

$4pG\ \rho\ (R,z) = \nabla^2\Phi(R,z) = \partial^2\Phi/\partial z^2 + (1/R)\partial/\partial R(R\partial\Phi/\partial P)$

Motion Perpendicular to the Plane- Alternate Analysis-(S+G pgs140-145, MBW pg 163)

the first derivative of the potential is the grav force perpendicular to the plane - call it K(z)

$n_*(z)$ is the density of the tracer population and
$\sigma_z(z)$ is its velocity dispersion

then the 1-D Poisson's eq $4\pi G\rho_{tot}(z,R)=d^2\phi(z,R)/dz^2$
where ρ_{tot} is the total mass density - put it all together to get

$4\pi G\rho_{tot}(z,R)=-dK(z)/dz$ (S+G 3.93)

$d/dz[n_*(z)\sigma_z(z)^2]=n_*(z)K(z)$ - to get the data to solve this have to determine $n_*(z)$ and $\sigma_z(z)$ for the tracer populations(s)

Use of Jeans Eqs

- Surface mass density near sun
- Poissons eq $\nabla^2\phi=4\pi\rho G=-\nabla\bullet\mathbf{F}$
- Use cylindrical coordinates
- $(1/R)\partial/\partial R(RF_R) + \partial F_z/\partial z=-4\pi\rho G$

- $F_R=-v_c/R$ v_c circular velocity (roughly constant near sun) -

(F_R force in R direction) so

$\partial F_R/\partial R=(-1/4\pi G)\partial F_z/\partial z$; **only vertical gradients count**

since the surface mass density $\Sigma=2\int\rho dz=-F_z/2\pi G$

(integrate 0 to $+\infty$ thru plane)

Now use Jeans eq: $nF_z-\partial(n\sigma^2_z)/\partial z+(1/R)\partial/\partial R(Rn\sigma^2_{zR})$; if R+z are separable

e.g $\phi(R,z)=\phi(R)+\phi(z)$ then $\sigma^2_{zR}\sim 0$ and voila! (eq 3.94 in S+G)

$\Sigma=1/2\pi Gn\ \partial(n\sigma^2_z)/\partial z$; need to observe the number density distribution of some tracer of the potential above the plane and its velocity dispersion distribution perpendicular to the plane goes at n $\exp(-z/z_0)$

Use of Jeans Eq For Galactic Dynamics

- Accelerations in the z direction from the Sloan digital sky survey for

1) all matter

2) ʻknown' baryons only

ratio of the 2 (bottom panel

use this data + Jeans eq (see below, to get the total acceleration

(in eqs ν is the density of tracers, v_ϕ is the azimuthal velocity (rotation)

$$a_R = \sigma_{RR}^2 \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma_{RR}^2}{\partial R} + \sigma_{RZ}^2 \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma_{RZ}^2}{\partial Z} + \frac{\sigma_{RR}^2}{R} - \frac{\sigma_{\phi\phi}^2}{R} - \frac{\overline{v_\phi}^2}{R}, \quad (1)$$

$$a_Z = \sigma_{RZ}^2 \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma_{RZ}^2}{\partial R} + \sigma_{ZZ}^2 \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma_{ZZ}^2}{\partial Z} + \frac{\sigma_{RZ}^2}{R}. \quad (2)$$

Given accelerations $a_R(R, Z)$ and $a_Z(R, Z)$, *i.e.* the gradient of the gravitational potential, the dark matter contribution can be estimated after accounting for the contribution from visible matter.

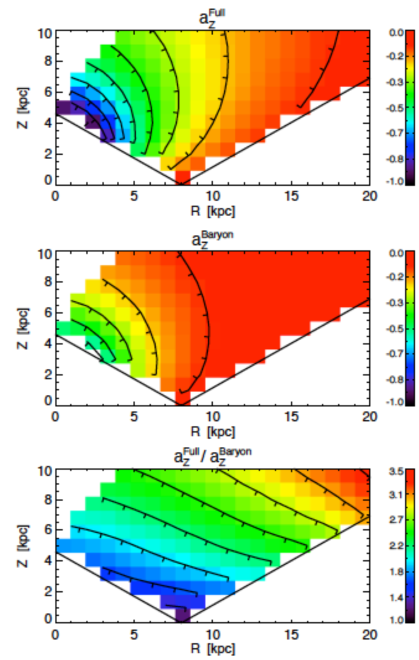

Figure 1. A comparison of the acceleration in the Z dir when all contributions are included (star, gas, and dark particles; top panel) to the result without dark matter (panel). The acceleration is expressed in units of 2.9×10^{-13} k The ratio of the two maps is shown in the bottom panel. importance of the dark matter increases with the distance fro origin; at the edge of the volume probed by SDSS ($R \sim 2$

Results

- Using dynamical data and velocity data get estimate of surface mass density in MW

Σ_{total}~70 +/- 6$M_\odot/pc^2$

Σ_{disk}~48+/-9 $M_\odot/pc^2$

Σ_{star}~35$M_\odot/pc^2$

Σ_{gas}~13$M_\odot/pc^2$

we know that there is very little light in the halo so direct evidence for dark matter

What Does One Expect The Data To Look Like

- A full-up numerical simulation from cosmological conditions of a MW like galaxy-this 'predicts' what a_Z should be near the sun (Loebman et al 2012)
- **Notice that it is not smooth or monotonic and the the simulation is neither perfectly rotationally symmetric nor steady state..**
- errors are on the order of 20-30%- figure shows comparison of true radial and z accelerations compared to Jeans model fits

1 kpc x 1kpc bins; acceleration units of $2.9\text{x}10^{-13}$ km/sec^2