Dynamics and how to use the orbits of stars to do interesting things

chapter 3 of S+G- very hard to summarize in a lecture; *please read it carefully*

This lecture will be a bit chaotic

A Reminder of Newtonian Physics

Newtons law of gravity tells us that two masses attract each other with a force

$$\frac{d}{dt}(m\boldsymbol{v}) = -\frac{GmM}{r^3}\boldsymbol{r}$$

If we have a collection of masses acting on a mass m_α the force is

$$\frac{d}{dt}(m_\alpha \boldsymbol{v}_\alpha) = -\sum_\beta \frac{G m_\alpha M_\beta}{|\boldsymbol{x}_\alpha - x_\beta|^3}(\boldsymbol{x}_\alpha - x_\beta)\,,\ \alpha \neq \beta$$

$$\frac{d}{dt}(m\boldsymbol{v}) = -m\nabla\Phi(\boldsymbol{x}),$$

with

$$\Phi(\boldsymbol{x}) = -\sum_\alpha \frac{G m_\alpha}{|\boldsymbol{x} - x_\alpha|},\ for\ \boldsymbol{x} \neq \boldsymbol{x}_\alpha$$

the gravitational potential. If we can approximate the discrete stellar distribution with a continuous distribution ρ.

$$\Phi(\boldsymbol{x}) = -\int \frac{G\rho(\boldsymbol{x}')}{|\boldsymbol{x} - x'|} d^3\boldsymbol{x}'$$

Conservation of Energy and Angular Momentum

In the absence of external forces a star will conserve energy along its orbit

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v})=-m\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}),$$

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v})+m\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x})=0$$

But since $\frac{d\Phi}{dt}=\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x})$

$$\frac{d}{dt}\left[\frac{m}{2}(\boldsymbol{v}^2)+m\Phi(\boldsymbol{x})\right]=0$$

This is just the KE + PE

Angular momentum L

$$\frac{dL}{dt}=\boldsymbol{x}\times m\frac{d\boldsymbol{v}}{dt}=-m\boldsymbol{x}\times\nabla\Phi$$

Some Basics - M. Whittle

- The gravitational potential energy is a scalar field
- its gradient gives the net gravitational force (per unit mass) which is a vector field :

$$\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} \, d^3\mathbf{r}'$$

$$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = G \int_V \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} \, \rho(\mathbf{r}') \, d^3\mathbf{r}'$$

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = -4\pi G\rho(\mathbf{r})$$

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G\rho(\mathbf{r})$$

Poissons eq inside the mass distribution

$$\nabla^2\Phi(\mathbf{r}) = 0$$

Outside the box

Poisson's Eq+ Definition of Potential Energy (W)

So the force per unit mass is

$$\boldsymbol{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = \int G\,\rho(\mathbf{x}')\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}|^3}d^3\mathbf{x}'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides S+G pg 112-113

$$\nabla^2\Phi(\mathbf{x}) = -\nabla^2\int\frac{G\,\rho(\mathbf{x}')}{|\mathbf{x}-x'|}d^3\mathbf{x}'$$

$$= 4\pi G\,\rho(\mathbf{x})$$ Poisson's equation.

$$W = \frac{1}{2}\int_V \rho(\mathbf{r})\,\Phi(\mathbf{r})\,d^3\mathbf{r} = -\frac{1}{8\pi G}\int_V |\nabla\Phi|^2\,d^3\mathbf{r}$$

Escape Speed/Angular Momentum Changes

- As r goes to infiinity f(r) goes to zero
- so to escape $v^2 > 2\phi(r)$; e.q. v_{esc}=sqrt($-2\phi(r)$)

Gravity and Dynamics

- Newton's theorem sec 3.1
- Gravitational force at a point outside a closed **sphere** is the same as if all the mass were at the center
- Potential due to mass ϕ=-GM/r = G$\int\rho d\Omega$/R
- This does not work for a thin disk- cannot ignore what is outside of a given radius
- the general solution is quite messy and involves Bessel functions and Hankel transforms (see later)

- One of the prime observables (especially for spirals) is the circular velocity; in general it is $V^2(R)/R=G(M<R)/R^2$ more accurate estimates need to know shape of potential
- so one can derive the mass of a flattened system from the rotation curve

Analogy of Stellar Systems to Gases
- Discussion due to Mark Whittle

- similarities :

comprise many, interacting objects which act as points (separation >> size)

can be described by distributions in space and velocity eg Maxwellian velocity distributions; uniform density; spherically concentrated etc.

Stars or atoms are neither created nor destroyed -- they both obey continuity equations

All interactions as well as the system as a whole obeys conservation laws (eg energy, momentum)if isolated

- But :
- The relative importance of short and long range forces is radically different :
  - atoms interact only with their neighbors, however
  - stars interact continuously with the entire ensemble via the long range attractive force of gravity
- eg uniform medium : $F \sim G\,\Omega\rho / r^2 \;\sim\; r^2 dr\ r^2 \;\sim\; \rho\, dr$ **~equal force from all distances**

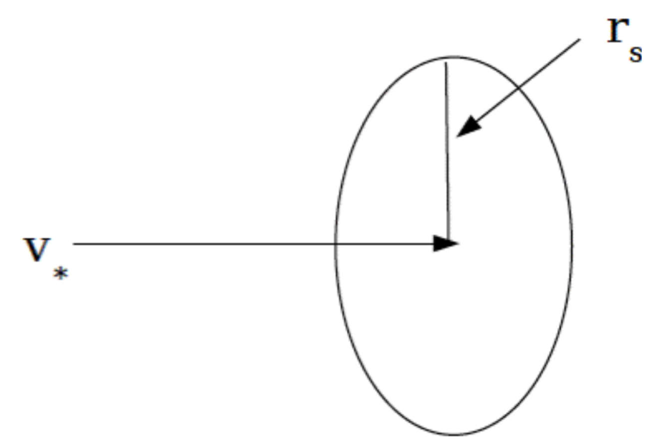
Analogy of Stellar Systems to Gases
- Discussion due to Mark Whittle

- The relative frequency of strong encounters is radically different :
- -- for atoms, encounters are frequent and all are strong (ie $\delta V \sim V$)
- -- for stars, pairwise encounters are very rare, and the stars move in the smooth global potential (e.g. S+G 3.2)

- Consequently, given the parallels between gas (fluid) dynamics and stellar dynamics many of the same equations can be used as well as :
- ---> concepts such as Temperature and Pressure can be applied to stellar systems
- ---> we use analogs to the equations of fluid dynamics and hydrostatics
- there are also some interesting differences
- ---> pressures in stellar systems can be anisotropic
- ---> stellar systems have negative specific heat and evolve away from uniform temperature.

How Often Do Stars Encounter Each Other

For a 'strong' encounter $GmM/r>1/2mv^2$ e.g. potential energy exceeds KE
So a critical radius is $r<r_s=2GM/v^2$



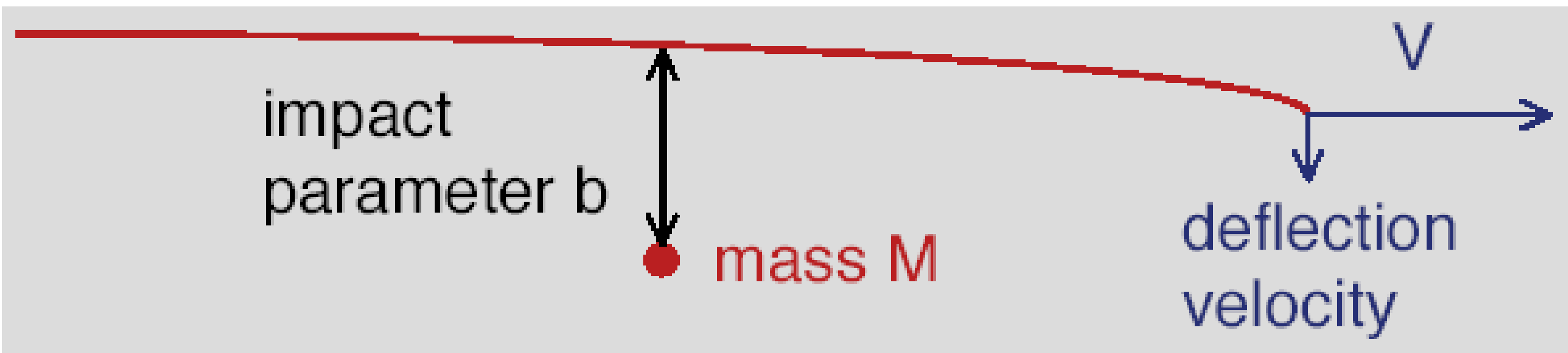
Putting in some typical numbers $m\sim 1/2M_\odot$
$v=30$km/sec $r_s=1$AU
So how often do stars get that close?

consider a cynlinder $Vol=\pi r^2{}_s vt$; if have n stars per unit volume than on average the encounter occurs when
$n\pi r^2{}_s vt=1$, $t_s=v^3/\ 4\pi nG^2m^3$

Putting in typical numbers
$=4\times10^{12}(v/10\text{km/sec})^3(m/M_\odot)^{-2}(n/pc3)^{-1}$ yr- a very long time (universe is only 10^{10}yrs old- galaxies are essentially collisionless

What About Collective Effects ? sec 3.2.2

For a weak encounter b >> r_s
Need to sum over individual interactions- **effects are also small**



Time Scales for Collisions

- N particles of radius $r_{p;}$ Cross section for a direction collision $\sigma_d = \pi r^2_p$
- Definition of mean free path; if V is the volume of a particle $4/3\pi r^3_p$

  $\lambda = V/n\sigma_d$ where n is the number density of particles (particles per unit volume) $n = 3N/4\pi r^3_p$

  and the characteristic time between collisions (Dim analysis)is $t_{collision} = \lambda/v \sim (\ell/r_p)^2 t_{cros}/N$ where v is the velocity of the particle.

  for a body of size ℓ, $t_{cross} = \ell/v$

  So lets consider a galaxy with l~10kpc, N=10^10 stars and v~200km/sec

  if r_p = Rsun $t_{collision} \sim 10^{21}$ yrs

- For indirect collisions the argument is more complex (seeS+G sec 3.2.2 MWB pg 231) but the answer is the same - it takes a very long time for star interactions to exchange energy (relaxation).
- $t_{relax} \sim N t_{cross}/10 \ln N$
- Its only in the centers of the densest globular clusters and galactic nuclei that this is important

So Why Are Stars in Rough Equilibrium

- In order to use the machinery developed in the text sec 3.4
- It seems that another process 'violent relaxation' (BW pg 251) is crucial.
- This is due to rapid change in the gravitational potential.
- Stellar dynamics describes in a statistical way the collective motions of stars subject to their mutual gravity-The essential difference from celestial mechanics is that each star contributes more or less equally to the total gravitational field, whereas in celestial mechanics the pull of a massive body dominates any satellite orbits
- The long range of gravity and the slow "relaxation" of stellar systems prevents the use of the methods of statistical physics as stellar dynamical orbits tend to be much more irregular and chaotic than celestial mechanical orbits-....woops.

Full Up Equations of Motion- Stars as an Ideal Fluid(S+G pgs140-144, MBW pg 163)

Continuity equation (particles not created or destroyed)

$d\rho/dt+\rho\nabla\cdot v=0$; $d\rho/dt+d(\rho v)/dr=0$

Eq's of motion (Eulers eq)
$dv/dr=\nabla P/\rho-\nabla\Phi$

Poissons eq

Simple/Complex

- Most often simple analytic forms for the density distribution or potential lead to complex analytic forms for the other.
- Point mass $\phi(r)=-GM/r$; $F(R)=-\nabla\phi=d\phi/dr=)=-GM/r^2$

- Shell:use Newton's theorem $\phi(r)$

outside like point mass, inside constant

$F(R)=0$

- Homogenous sphere radius a, constant density

outside like point mass

inside $\phi(r)=-2\pi\rho G(a^2-r^2/3)$;$F(R)=4/3\pi\rho Gr$

circular velocity $V_c=(4/3\pi\rho G)^{1/2}r$ so angular velocity $\omega(r)$ is constant

spiral galaxy rotation curves have constant $V_{c;}$

so $V_0^2/R=F(R)=-\phi(r)=-V_0^2\ln R+\text{constant}$

- Singular isothermal sphere $\rho(r)=\rho_0(r/a)^{-2}$ $V_{c;}=\text{sqrt}(4\pi\rho_0 Ga^2)$ constant at all radii; $\phi(r)=-4\pi\rho(0)Ga^2\ln(r/a)$

Rotation Curve Mass Estimates

- sec sec 11.1.2 in MBW
- Galaxy consists of a axisysmmetric disk and spherical dark matter halo
- Balance centrifugal force and gravity
- $V^2(R)=R\mathbf{F}(R)$; $F(R)$ is the acceleration in the disk
- Split rotation into 2 parts due to disk and halo

  $V^2(R)=V^2_{,d}(R)+V^2_h(R)$
- for a **spherical** system

  $V^2(R)=rd\phi/dr=GM(r)/r$
- Few analytic solutions: point mass $V_c(R)\sim r^{-1/2}$

  singular isothermal sphere $V_c(R)$=constant (see S+G eq 3.14)

  uniform sphere $V_c(R)\sim r$

for a pseudo-isothermal (S+G problem 2.20)

$\rho(r)=\rho(0)(R^2_c/R^2+R^2_c)$; $\rho(0)= V(\infty)^2/4\pi G\, R^2_c$ and the velocity profile is

$V(R)^2=V(\infty)^2(1-R_c/R \tan^{-1} R/R_c)$;for a **NFW** potential get a rather messy formula

Rotation Curve Mass Estimates

For the **disk radial component**: $V^2_c(R)=R(\partial\phi/\partial r)$ at $z=0$

Perpendicular to the disk $(\partial\phi/\partial z)=-2\pi G\Sigma$ above disk; inside disk at displace from midpalne z; $2\pi G\rho_0 z$

the potential for a thin disk is the Bessel function (next page)

Disks are Messy

- Skipping the integrals of Bessel functions (eq 11.2 MBW) one gets
- $V^2_{c,d}(R) = 4\pi G\Sigma_0 R_d y^2[I_o(y)K_0(y) - I_1(y)K_1(y)]$

$y=R/(2R_d)$ and I and K are Bessel functions of the first and second kinds:which do not have simple asymptotic forms

Important bits: $V^2_{c,d}(R)$ depends only on radial scale length R_d and its central surface density Σ_0

Radial scale length of a spiral disk

$\Sigma(r)=\Sigma_0 \exp(-R/R_d)$**;** integrate over r to get total mass $M_d=2\pi\Sigma_0 R^2_d$

Vertical density distribution is also an exponential $\exp(-z/z_0)$ so total distribution is product of the two

$\rho(R,z)=\rho_0 \exp(-R/R_d)\exp(-z/z_0)$

while we may know the scale length of the stars, that of the dark matter is not known.

Also the nature of the dark matter halo is not known:- disk/halo degeneracy

Motion Perpendicular to the Plane-(S+G pgs140-144, MBW pg 163)

For the motion of stars in the vertical direction only
$d/dz[n_*(z)\sigma_z(z)^2]=-n_*(z)d\phi(z,R)/dz$; where **$\phi(z,R)$is the vertical grav potential- independent of motion in the disk plane (eq. 3.68)**
the first derivative of the potential is the grav force perpendicular to the plane - call it K(z)
$n_*(z)$ is the density of the tracer population and
$\sigma_z(z)$ is its velocity dispersion

then the 1-D Poissons eq $4\pi G\rho_{tot}(z,R)=d^2\phi(z,R)/dz^2$
where ρ_{tot} is the total mass density - put it all together to get

$4\pi G\rho_{tot}(z,R)=-d\,K(z)/dz$ (S+G 3.93)
S+G 3.95 potential above uniform sheet is $\phi(x)=2\pi G\Sigma|z|$
$d/dz[n_*(z)\sigma_z(z)^2]=n_*(z)K(z)$ - to get the data to solve this have to determine $n_*(z)$ and $\sigma_z(z)$ for the tracer populations(s)

- Separate potential into r and z parts

$\phi(r,z)=B(R)Z(z)$

outside disk $\nabla^2\phi=0$; find $Z(z)=A\exp(k|z|)$

eq for R dependence of potential is

$(1/R)(d/dR(RdB/dr)+k^2B(R)=0$- the solutions of this are Bessel functions J(r); but it gets even messier

Important result

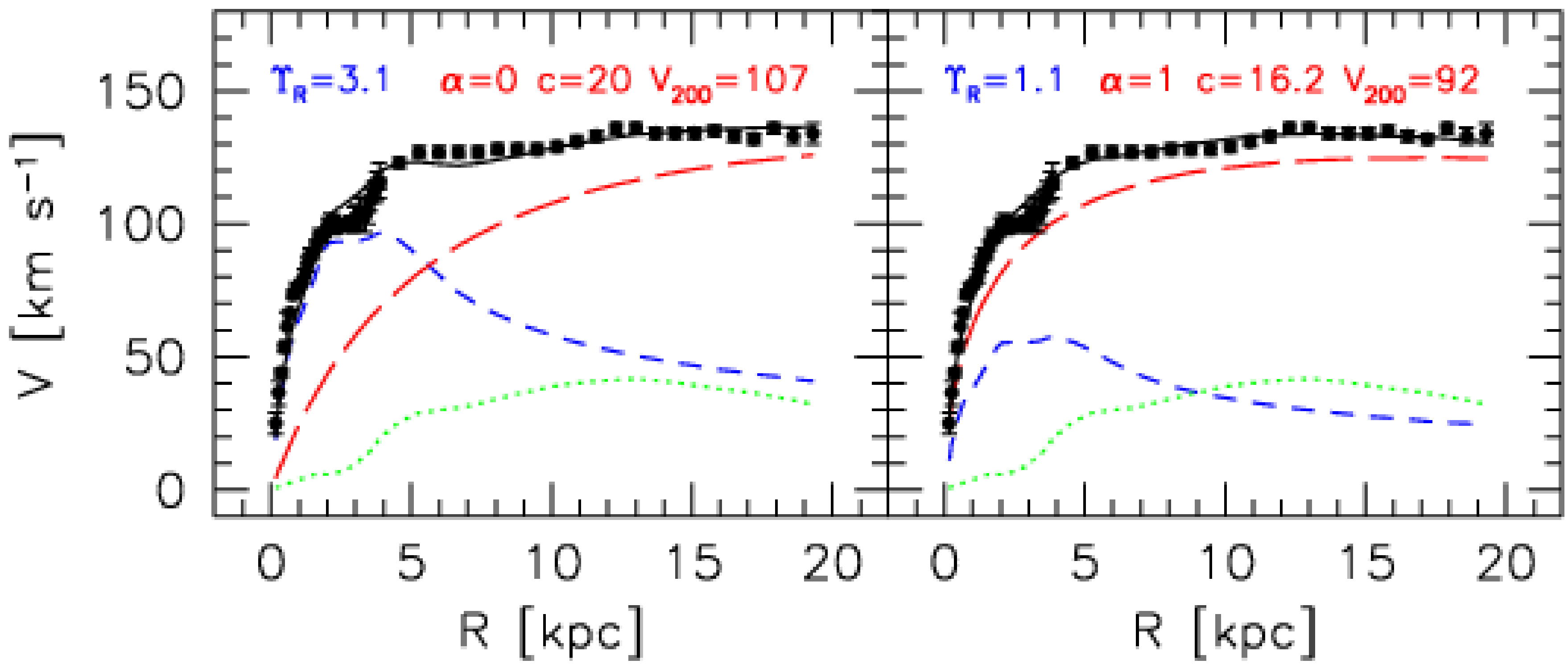
- $Rd\phi/dR=v^2_c = GM(R)/R$ to within 10% for most 'reasonable' forms of mass distribution

see http://www.ast.cam.ac.uk/~ccrowe/Teaching/Handouts for lots of derivations/

Disk Halo Degeneracy

- MBW fig 11.1: two solutions to rotation curve of NGC2403: stellar disk (blue lines), dark matter halo - red lines.

- Left panel is a 'maximal' disk, using the highest reasonable mass to light ratio for the stars, the right panel a lower value of M/L



Virial Theorem

- This fundamental result describes how the total energy (E) of a self-gravitating system is shared between kinetic energy and potential energy .
- Go to one dimension and assume steady state

$$\frac{\partial}{\partial x}\left[\rho v^{2} \right] + \rho \frac{\partial \Phi}{\partial x} = 0$$

Integrate over velocity and space and one finds

$-2E_{kinetic}=PE_{potential}$
see text pgs 120-121 for full derivation

This is important for find the masses of systems whose orbital distribution
is unknown or very complex and more or less in steady state (so assumptions in derivation are ok)

Virial Theorem

- S+G pg 120-121
- A rather different derivation (due to H Rix)
- Consider (for simplicity) the 1-D Jeans eq in steady state (see later)
- $\partial/\partial x[\rho v^2]+\rho\partial\phi/\partial x=o$
- Integrate over velocities and then over positions...
- $-2E_{kin}=E_{pot}$

- or restating in terms of forces
- if T= total KE of system of N particles < >= time average
- $2\langle T\rangle=-\Sigma(F_k \bullet r_k)$; summation over all particles k=1,N

call the 'virial 'Q

$$Q= \frac{1}{2}\frac{dI}{dt} = m\sum r.\frac{dr}{dt} = \sum p.r$$

$$dQ/dt= \sum F.r + 2T$$

Virial Theorem

- Another derivation following Bothun 4.1.1
- Moment of inertia, I, of a system of N particles
- $I=\Sigma m_i r_i^2$ sum over i=1,N (express r_i^2 as $(x_i^2+y_i^2+z_i^2)$)
- take the first and second time derivatives ; let dx^2/dt^2 be symbolized by ***x,y,z***
- $dI^2/dt^2 = \Sigma m_i (dx_i^2/dt+dy_i^2/dt+dz_i^2/dt)+\Sigma m_i(x_i \boldsymbol{x}+y_i \boldsymbol{y}+z_i \boldsymbol{z})$

mv^2 (KE)+Potential energy (W) r •(m**a**)

after a few dynamical times, if unperturbed a system will come into Virial equilibrium–time averaged inertia will not change so 2<T>+W=0

For self gravitating systems $W=-GM^2/2R_H$; is the harmonic radius- the sum of the distribution of particles appropriately weigthed

$1/R_H =1/N\ \Sigma_i\ 1/r_i$

The virial mass estimator is $M=2\sigma^2 R_H/G$; for many mass distributions $R_H \sim 1.25\ R_{eff}$

where sR_{eff} is the half light radius σ is the 3-d velocity dispersion

Virial Theorem

- This fundamental result describes how the total energy (E) of a self-gravitating system is shared between kinetic energy and potential energy in a TIME AVERAGED sense.
- S+G pg 122 show that, in general $PE=GMm/2\varepsilon r_c$ where e is a function of the shape of the potential and r_c is some characteristic length (often the half light radius)

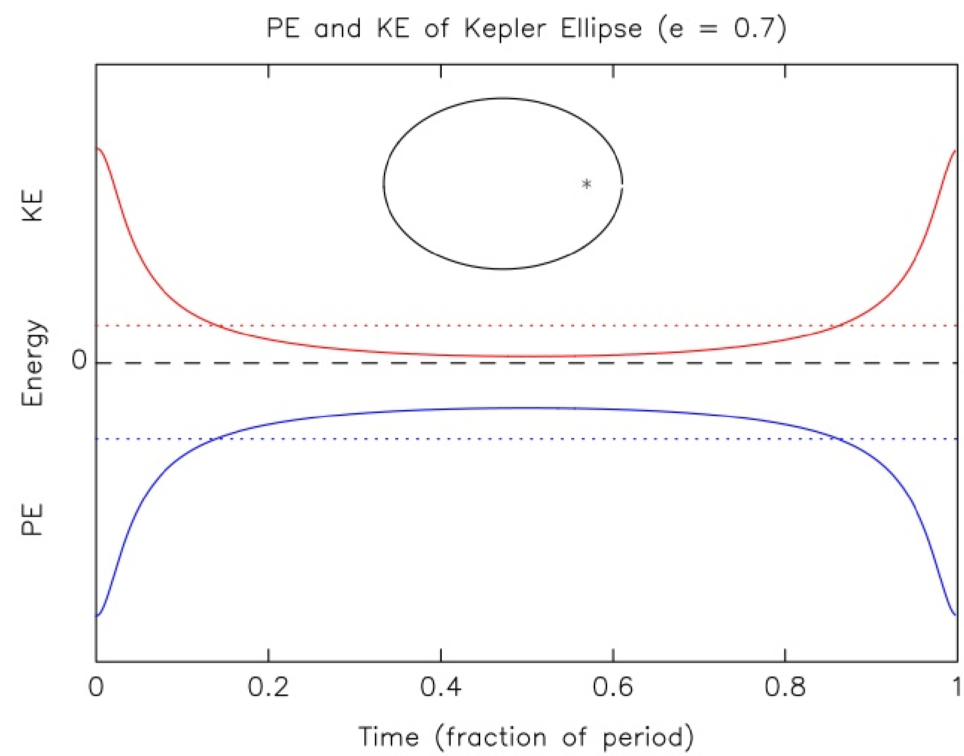
Special cases

Circular $mv^2/r=GMm/r^2$

multiply by r to get

$mv^2=GMm/r$ or $2KE=-PE$

Keplerian orbit: ratio of potential to kinetic changes over orbit, but when average over orbit satisfies virial theorem



Use of Virial Theorem

- Consider a statistically steady state, spherical, self gravitating system of N objects with average mass m and velocity dispersion σ.
- Total KE=$(1/2)Nm\sigma^2$
- If average separation is r the PE of the system is $U=(-1/2)N(N-1)Gm^2/r$
- Virial theorem E=-U/2 so the total mass is $M=Nm=2\pi\sigma^2/G$ or using L as light and Σ as surface light density

  $\sigma^2 \sim (M/L)\Sigma R$-picking a scale (e.g. half light radius R_e)

  $R_e \sim \sigma^\alpha \Sigma^\beta$ $\alpha=2, \beta=1$ from viral theorem

  value of proportionality constants depends on shape of potential

  For clusters of galaxies and globular clusters often the observables are the light distribution and velocity dispersion. then one measures the ratio of mass to light as

  $M/L \sim 9\sigma^2/2\pi G I(0) r_c$ for spherical isothermal systems

Use of Virial Theorem

- Motion of a star in an external potential: e.g how do stars in galaxies move!

divide by mass and get

$<v^2> = -<\nabla\phi(x)\bullet x>$; circular orbit in a spherically symmetric potential equivalent to

$v^2(r) = -GM(<r)/r$

Collisionless Boltzmann Eq

- When considering the structure of galaxies cannot follow each individual star (10^{11} of them!),

- Consider instead stellar density and velocity distributions. However, a fluid model not really appropriate since a fluid element has a single velocity, which is maintained by particle-particle collisions on a scale much smaller than the element. For stars in the galaxy this **not true**-stellar collisions are very rare, and even encounters where the gravitational field of an individual star is important in determining the motion of another are very infrequent
- So taking this to its limit, treat each

  particle as being collisionless, moving under the influence of the mean potential generated by all the other particles in the system $\phi(x,t)$

Collisionless Boltzmann Eq

- The distribution function is defined such that $f(\mathbf{r},\mathbf{v},t)d^3\mathbf{x}d^3\mathbf{v}$ specifies the number of stars inside the volume of phase space $d^3\mathbf{x}d^3\mathbf{v}$ centered on $(\mathbf{x},\mathbf{v})$ at time t- an equivalent statement is

At time t a full description of the state of this system is given by specifying the number of stars $f(x, v, t)d^3xd^3v$

Then $f(\mathbf{x}, \mathbf{v}, t)$ is called the "distribution function" (or "phase space density") of the system.

$f \geq 0$ since no negative star densities

For a collisionless stellar system in dynamic equilibrium, the gravitational potential, ϕ , relates to the phase-space distribution of stellar tracers $f(\mathbf{x}, \mathbf{v}, t)$, via the collisionless Boltzmann Equation

Collisionless Boltzmann Eq

- This results in (S+G pg 143)

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0,}$$

- the flow of stellar phase points through phase space is incompressible – the phase-space density of points around a given star is always the same

- The distribution function f is a function of seven variables, so solving the collisionless Boltzmann equation in general is hard. So need either simplifying assumptions (usually symmetry), or try to get insights by taking moments of the equation.

define n(x,t) as the number density of stars at position x then the first moment is $\partial n/\partial t+\partial/\partial x(nv)=0$; the continuity equation of a fluid

second moment
$n\partial v/dt+nv\partial v/dx=-n\partial \phi/\partial x-\partial/\partial x(n\sigma)$
σ is the velocity dispersion
But unlike fluids do not have thermodynamics to help out.... nice math but not clear how useful

Collisionless Boltzmann Eq

- .

astronomical structural and kinematic observations provide information only about the **projections of phase space distributions along lines of sight**, limiting knowledge about f and hence also about ϕ.

Therefore all efforts to translate existing data sets into constraints on involve simplifying assumptions.

• dynamic equilibrium,
•spherical symmetry
• particular functional forms for the distribution function and/or the gravitational potential.

Jeans Equations

- Since f is a function of 7 variables obtaining a solution is challenging
- Take moments (e.g. integrate over **v**)
- let n be the space density of 'stars'

$\partial n/\partial t+\partial(n\langle v_i\rangle)/\partial x_i=0$

$\partial(n\langle v_j\rangle/\partial t)+\partial(n\langle v_i v_i\rangle)/\partial x_i+n\partial\phi/\partial x_j=0$

$n\partial(\langle v_j\rangle/\partial t)+n\langle v_i\rangle\,\partial\langle v_j\rangle/\partial x_i=-n\partial\phi/\partial x_j-\partial(n\sigma^2_{ij})/\partial x_i$

so n is the integral over velocity of f; $n=\int f\, d^3v$

$\langle v_i\rangle$ is the mean velocity in the i^{th} direction $=1/n\int f\, v_i\, d^3v$

the term $-\partial(n\sigma^2_{ij})/\partial x_i$ is like a pressure, but allows for different pressures in different directions- important in elliptical galaxies and bulges 'pressure supported' systems

Jeans Equations Another Formulation

- Jeans equations follow from the collisionless Boltzmann equation; Binney & Tremaine (1987), MWB 5.4.2. S+G sec 3.4 .

cylindrical coordinates and assuming an axi-symmetric and steady-state system, the accelerations in the radial (R) and vertical (Z) directions can be expressed in terms of observable quantities:

the stellar number density distribution ν_*

and 4 velocity components

a rotational velocity v_ϕ

and 4 components of random velocities

(velocity dispersion components)

$\sigma_{\phi\phi}, \sigma_{RR}, \sigma_{ZZ}, \sigma_{RZ}$

$$a_R = \sigma^2_{RR} \times \frac{\partial(\ln \nu)}{\partial R} + \frac{\partial \sigma^2_{RR}}{\partial R} + \sigma^2_{RZ} \times \frac{\partial(\ln \nu)}{\partial Z} + \frac{\partial \sigma^2_{RZ}}{\partial Z} + \frac{\sigma^2_{RR}}{R} - \frac{\sigma^2_{\phi\phi}}{R} - \frac{\overline{v_\phi}^2}{R},$$

$$a_Z = \sigma^2_{RZ} \times \frac{\partial(\ln \nu)}{\partial R} + \frac{\partial \sigma^2_{RZ}}{\partial R} + \sigma^2_{ZZ} \times \frac{\partial(\ln \nu)}{\partial Z} + \frac{\partial \sigma^2_{ZZ}}{\partial Z} + \frac{\sigma^2_{RZ}}{R}.$$

where a_Z, a_R are accelerations in the appropriate directions-

given these values (which are the gradient of the gravitational potential), the dark matter contribution can be estimated after accounting for the contribution from visible matter

Use of Jeans Eqs

- Surface mass density near sun
- Poissons eq $\nabla^2\phi=4\pi\rho G=-\nabla\bullet\mathbf{F}$
- Use cylindrical coordinates
- $(1/R)\partial/\partial R(RF_R) + \partial F_z/\partial z=-4\pi\rho G$

- $F_R=-v_c/R$ v_c circular velocity (roughly constant near sun) so

(F_R force in R direction) so

$\partial F_R/\partial R=(-1/4\pi G)\partial F_z/\partial z$; **only vertical gradients count**

since the surface mass density $\Sigma=2\int\rho dz=-F_z/2\pi G$

(integrate 0 to $+\infty$ thru plane)

Now use Jeans eq: $nF_z-\partial(n\sigma^2_z)/\partial z+(1/R)\partial/\partial R(Rn\sigma^2_{zR})$; if R+z are separable

e.g $\phi(R,z)=\phi(R)+\phi(z)$ then $\sigma^2_{zR}\sim0$ and voila! (eq 3.94 in S+G)

$\Sigma=1/2\pi Gn\,\partial(n\sigma^2_z)/\partial z$; need to observe the number density distribution of some tracer of the potential above the plane and its velocity dispersion distribution perpendicular to the plane goes at n $\exp(-z/z_0)$

Jeans Continued

- Using dynamical data and velocity data get estimate of surface mass density

$\Sigma_{total} \sim 70 \pm 6 M_\odot/pc^2$

$\Sigma_{disk} \sim 48 \pm 9\ M_\odot/pc^2$

$\Sigma_{star} \sim 35 M_\odot/pc^2$

$\Sigma_{gas} \sim 13 M_\odot/pc^2$

we know that there is very little light in the halo so direct evidence for dark matter

Spherical system

- For a spherical system the Jeans equations simplify to (Walker 2012)

$$\frac{1}{\nu}\frac{d}{dr}\left(\nu\langle v_r^2\rangle\right)+2\frac{\beta_a\langle v_r^2\rangle}{r}=-\frac{GM(r)}{r^2},$$

where $\nu(r)$, $\langle v_r^2\rangle(r)$, and $\beta_a(r)\equiv 1-\langle v_\theta^2\rangle/\langle v_r^2\rangle$ describe the 3-dimensional density, radial velocity dispersion, and orbital anisotropy, respectively, of the (stellar) tracer component.

This will become very important for elliptical galaxies

Jeans Again

- $M_J = 1/8(\pi k T/G\mu)^{3/2}\, \rho^{-1/2}$

- In astronomical units this is

$M_J = 0.32 M_\odot (T/10k)^{3/2} (m_H/\mu)^{3/2} (10^6 cm^{-3}/n_H)^{1/2}$

So for star formation in the cold molecular medium with $T \sim 10k$ and $n_H \sim 10^5$- $M_J \sim 2M$

The growth time for the Jeans instability is

$\tau_J = 1/sqrt(4\pi G\rho) = 2.3x10^4 yr(n_H/10^6 cm^{-3})^{-1/2}$

For pure free fall

$\tau_J = (3\pi/32G\rho)^{1/2} = 4.4x10^4 yr(n_H/10^6 cm^{-3})^{-1/2}$

Jeans growth rate about 1/2 the free fall time

τ_s time scale is the sound crossing time across the Jeans Length

$c_s = sqrt(kT/m_H\mu)$ $\lambda_J = (\pi c^2_s/G\rho)^{1/2}$ $\tau_s = \lambda_J/c_s$

Use of Jeans Eq Fr Galactic Dynamices

- Accelerations in the z direction fromthe Sloan digital sky survey for

1) all matter

2) ‘known' baryons only

ratio of the 2 (bottom panel

use this data + Jeans eq (see below, to get the total acceleration

(in eqs ν is the density of tracers, v_ϕ is the azimuthal velocity (rotation)

$$a_R = \sigma^2_{RR} \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma^2_{RR}}{\partial R} + \sigma^2_{RZ} \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma^2_{RZ}}{\partial Z} + \frac{\sigma^2_{RR}}{R} - \frac{\sigma^2_{\phi\phi}}{R} - \frac{\overline{v_\phi}^2}{R}, \quad (1)$$

$$a_Z = \sigma^2_{RZ} \times \frac{\partial(\ln\nu)}{\partial R} + \frac{\partial\sigma^2_{RZ}}{\partial R} + \sigma^2_{ZZ} \times \frac{\partial(\ln\nu)}{\partial Z} + \frac{\partial\sigma^2_{ZZ}}{\partial Z} + \frac{\sigma^2_{RZ}}{R}. \quad (2)$$

Given accelerations $a_R(R,Z)$ and $a_Z(R,Z)$, *i.e.* the gradient of the gravitational potential, the dark matter contribution can be estimated after accounting for the contribution from visible matter.

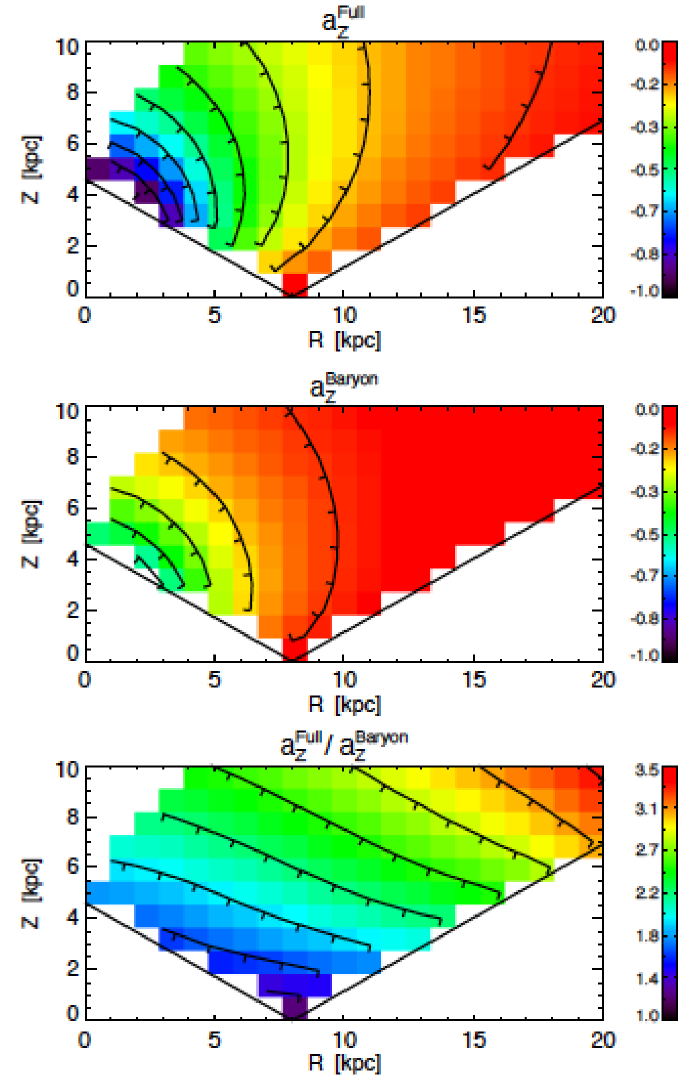


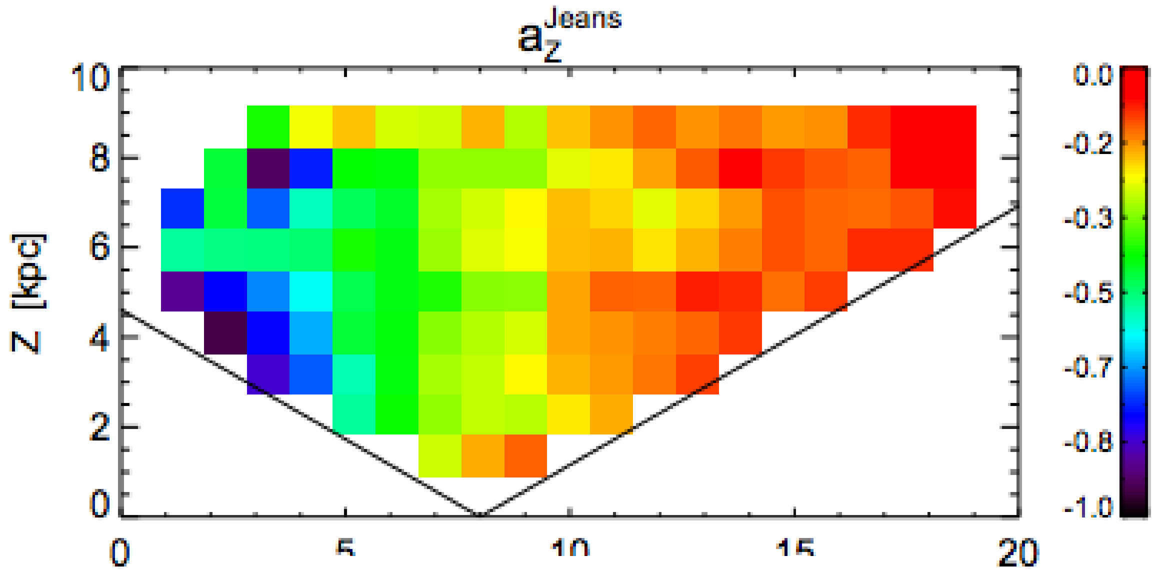
Figure 1. A comparison of the acceleration in the Z dir when all contributions are included (star, gas, and dark particles; top panel) to the result without dark matter (panel). The acceleration is expressed in units of 2.9×10^{-13} The ratio of the two maps is shown in the bottom panel. importance of the dark matter increases with the distance fr origin; at the edge of the volume probed by SDSS ($R \sim 2$

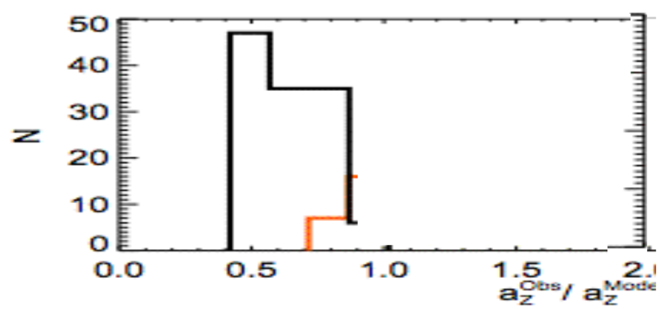
What Does One Expect The Data To Look Like

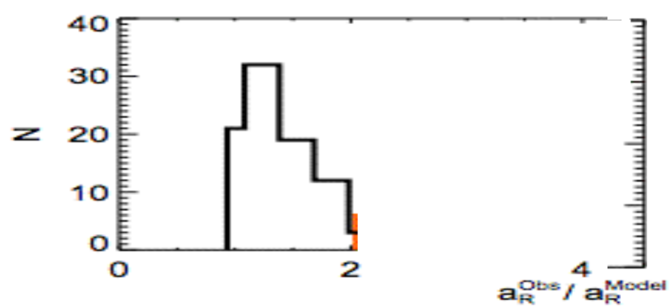
- A full-up numerical simulation from cosmological conditions of a MW like galaxy-this 'predicts' what a_Z should be near the sun (Loebman et al 2012)
- **Notice that it is not smooth or monotonic**

  and the the simulation is neither perfectly rotationally symmetric nor steady state..
- errors are on the order of 20-30%- figure shows comparison of true radial and z accelerations compared to Jeans model fits

1 kpc x 1kpc bins; acceleration units of 2.9×10^{-13} km/sec^2







Collisionless Boltzmann Eq

- Some simplifications
- $n\partial v/dt+nv\partial v/dx=-n\partial\phi/\partial x-\partial/\partial x(n\sigma)$
- assume isotropy, steady state, non-rotating
- then
- $-n\,\nabla\phi=\nabla n\sigma^2$
- using Poissons eq
- $\nabla^2\phi=4\pi\rho G$ and solve for $\sigma(r)^2$