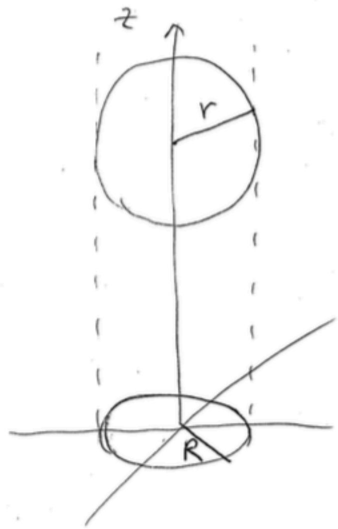
Project the spherical distribution onto a plane.
Because of the symmetry in the problem, choose any axis, say z, and use cylindrical polar coordinates.



$$2\pi R\,dR\,\Sigma(R) = 2\pi R\,dR \int_{-\infty}^{\infty} dz\, n(r)$$ (mass of an infinite annulus)

$$\Rightarrow \Sigma(R) = \int_{-\infty}^{\infty} n(r)\,dz = 2\int_{0}^{\infty} n(r)\,dz.$$

$r^2 = R^2 + z^2 \Rightarrow 2r\,dr = 2z\,dz$ ($dR = 0$ in the integration)

$$\Rightarrow dz = \frac{r}{z}\,dr = \frac{r}{\sqrt{r^2-R^2}}\,dr.$$

$$\therefore \Sigma(R) = 2\int_{R}^{\infty} \frac{n(r)\,r\,dr}{\sqrt{r^2-R^2}}$$

$\uparrow$ $r=R$ when $z=0$.

For $n(r) = n_0\left(\frac{r_0}{r}\right)^{\alpha}$,

$$\Sigma(R) = 2\int_{0}^{\infty} n_0 r_0^{\alpha}\frac{dz}{(R^2+z^2)^{\alpha/2}}.$$ Converges only for $\alpha > 1$

(limiting case $\alpha = 1$, as $z\to\infty$, $\Sigma(R) \to \int \frac{dz}{z}$ unbound

Density must fall off at least as $\propto \frac{1}{r}$ for the mass in the infinite annulus to be finite. $\alpha > 0$ also lets $\Sigma(R)$ remain finite as $R \to \infty$, $\Sigma(R)$ vanishes.

•

In spherical polar coordinates

$$\Sigma(R) = 2\int_R^\infty \frac{n_0\left(\frac{r_0}{r}\right)^\alpha r\,dr}{\sqrt{r^2-R^2}} = 2n_0\left(\frac{r_0}{R}\right)^\alpha \int_R^\infty \frac{\left(\frac{r}{R}\right)^{1-\alpha} dr}{\sqrt{\frac{r^2}{R^2}-1}}.$$

let $x = \frac{r}{R}$, $dr = R\,dx$

$$\Rightarrow \Sigma(R) = 2n_0 r_0 \left(\frac{r_0}{R}\right)^{\alpha-1} \int_1^\infty \frac{x^{1-\alpha}\,dx}{\sqrt{x^2-1}}$$

$$\Sigma(R=r_0) = 2n_0 r_0 \int_1^\infty \frac{x^{1-\alpha}\,dx}{\sqrt{x^2-1}}$$

$$\therefore\ \Sigma(R) = \Sigma(R=r_0)\left(\frac{r_0}{R}\right)^{\alpha-1}, \quad \underline{\alpha > 1}$$

Note that to be regular at $R=0$ $(r=0)$ require the mass element $4\pi r^2 n(r)\,dr$ to be finite as $r \to 0$.

$$4\pi r^2 n(r) = 4\pi r^2 n_0 \left(\frac{r_0}{r}\right)^\alpha \propto r^{2-\alpha}$$

Finite as $r \to 0$ for $\alpha \leq 2$.

$\boxed{6.5}$

$$f(q)\,dq = \frac{q\,dq}{\sqrt{1-(B/A)^2}\sqrt{q^2-(B/A)^2}}$$ is a probability distribution function.

Its integral $F(<q) \equiv \int_{B/A}^{q} f(q')\,dq'$ is a cumulative distribution function

They describe the probability distribution of projected axis ratio q for objects of true axis ratio B/A randomly oriented in space. Assuming they are randomly oriented in space, and that their true shape is described by two parameters then we can infer B/A from observed distribution of q.

$$F(<q) = \int_{B/A}^{q} \frac{q'\,dq'}{\sqrt{1-(B/A)^2}\sqrt{q'^2-(B/A)^2}}$$

$$= \frac{1}{\sqrt{1-(B/A)^2}}\left[\sqrt{q^2-(B/A)^2}\right]_{B/A}^{q}$$ by inspection

$$= \sqrt{\frac{q^2-(B/A)^2}{1-(B/A)^2}}$$ (spotting the anti-derivative of $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = \sqrt{f(x)}$)

= cumulative probability for $\frac{B}{A}$ to q.

By construction $F(<B/A)=0$ (can't be flatter than true ratio) and $F(<1)=1$. (can't be rounder than round)

By definition $F(a<q<b) = F(<b) - F(<a)$.

For objects with $\frac{B}{A}=0.8$ oriented randomly in space, expect to see

$$F(0.95<q<1) = F(<1) - F(<0.95)$$
$$= 1 - \sqrt{\frac{0.95^2-0.64}{1-0.64}} = 0.1461$$

$$F(0.8 < q < 0.85) = F(<0.85) - F(<0.8)$$

$$= \sqrt{\frac{0.85^2 - 0.64}{1 - 0.64}} - 0 = 0.4787$$

$$\frac{0.4787}{0.1461} = 3.3 .$$

Next part wants you to show that a smaller $\frac{B}{A}$ (a flatter oblate shape) has a higher fraction of

$$\frac{F(0.95 < q < 1)}{F(0.8 < q < 0.85)} = \sqrt{\frac{.95^2 - x^2}{1 - x^2}} \Big/ \left(\sqrt{\frac{.85^2 - x^2}{1 - x^2}} - \sqrt{\frac{.8^2 - x^2}{1 - x^2}} \right)$$

$$= \frac{\sqrt{.95^2 - x^2}}{\sqrt{.85^2 - x^2} - \sqrt{.8^2 - x^2}} \qquad x = \frac{B}{A}$$

Take the derivative w.r.t. x :

$$\frac{-x/\sqrt{.95^2 - x^2}}{\sqrt{.85^2 - x^2} - \sqrt{.8^2 - x^2}} - \frac{\sqrt{.95^2 - x^2}\left[-x\left(\frac{1}{\sqrt{.85^2 - x^2}} - \frac{1}{\sqrt{.8^2 - x^2}}\right)\right]}{\left(\sqrt{.85^2 - x^2} - \sqrt{.8^2 - x^2}\right)^2}$$ quotient rule.

Multiply out the terms in brackets in the second term,

$$\frac{-x/\sqrt{.95^2 - x^2} - x\sqrt{.95^2 - x^2} \Big/ \sqrt{.85^2 - x^2}\sqrt{.8^2 - x^2}}{\left(\sqrt{.85^2 - x^2} - \sqrt{.8^2 - x^2}\right)}, \quad \underline{\underline{< 0}} .$$

The important result is that this is negative

$\Rightarrow$ when $\frac{B}{A} \downarrow$, the ratio $\uparrow$.

For $-21 < M_B < -20$ there are ~ 126 points, 3 of which have $q > 0.95$. Since true $\frac{B}{A} \leq q$, it must be smaller than the smallest q observed (~ 0.6 in this luminosity range). The math just does not work out —— no oblate shape can give the observed distribution of q.

These galaxies are most likely triaxial, described by 3 parameters rather than two, and triaxial bodies are less likely to produce round projections.

So $F(<q)$ doesn't work out because it is not the correct description.

Shape of elliptical galaxies

Ellipticals have very little bulk rotation. Their shapes are primarily due to velocity dispersion anisotropy giving a triaxial shape. Bodies with high bulk angular momentum would separate into a spheroidal bulge and some disc component.

$\boxed{6.7}$ $KE = \frac{1}{2} M \langle v^2 \rangle$

Maxwellian velocities + isotropy

$\Rightarrow \langle v^2 \rangle = \sigma^2 = 3\sigma_r^2$

$\therefore \boxed{KE = \frac{3}{2} M \sigma_r^2 .}$

Given $GPE = -\frac{3GM^2}{5R_e}$, by virial theorem

$2 \times \frac{3}{2} M\sigma_r^2 - \frac{3GM^2}{5R_e} = 0 \Rightarrow \boxed{M = \frac{5\sigma^2 R_e}{G}}$

(6.1) $I(R) = I_e \exp\{-b[(\frac{R}{R_e})^{\frac{1}{n}} - 1]\} \propto \exp[-b(\frac{R}{R_e})^{\frac{1}{n}}]$.

$L = \int_0^\infty 2\pi R\, I(R)\, dR \propto I_e \int_0^\infty R \exp[-b(\frac{R}{R_e})^{\frac{1}{n}}]\, dR$

Extract the R_e dependence with substitution $x = \frac{R}{R_e}$

$\Rightarrow L \propto I_e R_e^2 \underbrace{\int_0^\infty x \cdot \exp[-b x^{1/n}]\, dx}_{\text{depends only on constants.}}$

$\therefore \boxed{L \propto I_e R_e^2}$

$\Rightarrow \boxed{\frac{M}{L} \propto \frac{\sigma^2 R_e}{I_e R_e^2} = \frac{\sigma^2}{I_e R_e}}$

(a) $\frac{M}{L}$, I_e constant $\Rightarrow$ $\sigma^2/R_e \propto$ constant. $\boxed{L \propto R_e^2 \propto \sigma^4}$

(b) (6.19) $R_e \propto \sigma^{6/5} I_e^{-4/5} \Rightarrow R_e^{-5/4} \sigma^{3/2} \propto I_e$.

$\frac{M}{L} \propto \frac{\sigma^2}{R_e} R_e^{5/4} \sigma^{-3/2} \propto \sigma^{1/2} R_e^{1/4} \propto \underline{\underline{M^{1/4}}}$