Dynamics and how to use the orbits of stars to do interesting things

chapter 3 of S+G- parts of ch 2 of B&T and parts of Ch 11 of MWB (Mo, van den Bosch, White)

A Guide to the Next Few Lectures

•The geometry of gravitational potentials : methods to derive gravitational potentials from mass distributions, and visa versa.

•Potentials define how stars move
consider stellar orbit shapes, and divide them into orbit classes.

•The gravitational field and stellar motion are interconnected :
the Virial Theorem relates the global potential energy and kinetic energy of the system.

• Collisions?

• The Distribution Function (DF) :
the DF specifies how stars are distributed throughout the system and with what velocities.

For collisionless systems, the DF is constrained by a continuity equation :
the Collisionless Boltzmann Equation

•This can be recast in more observational terms as the Jeans Equation.

The Jeans Theorem helps us choose DFs which are solutions to the continuity equations

*Adapted from M. Whittle

A Reminder of Newtonian Physics sec 2.1 in B&T

Newtons law of gravity tells us that two masses attract each other with a force

$$\frac{d}{dt}(m\boldsymbol{v}) = -\frac{GmM}{r^3}\boldsymbol{r}$$

ϕ (x) is the potential

If we have a collection of masses acting on a mass m_α the force is

$$\frac{d}{dt}(m_\alpha \boldsymbol{v}_\alpha) = -\sum_\beta \frac{Gm_\alpha M_\beta}{|\boldsymbol{x}_\alpha - x_\beta|^3}(\boldsymbol{x}_\alpha - x_\beta), \alpha \neq \beta$$

$$\frac{d}{dt}(m\boldsymbol{v}) = -m\nabla\Phi(\boldsymbol{x}),$$

with

$$\Phi(\boldsymbol{x}) = -\sum_\alpha \frac{Gm_\alpha}{|\boldsymbol{x} - x_\alpha|}, \ for\ \boldsymbol{x} \neq \boldsymbol{x}_\alpha$$

Gauss's thm $\int \nabla\phi \bullet ds^2 = 4\pi GM$
the Integral of the normal component over a closed surface =4πG x mass within that surface

the gravitational potential. If we can approximate the discrete stellar distribution with a continuous distribution ρ.

$$\Phi(\boldsymbol{x}) = -\int \frac{G\rho(\boldsymbol{x}')}{|\boldsymbol{x} - x'|} d^3\boldsymbol{x}'$$

3

Conservation of Energy and Angular Momentum

In the absence of external forces a star will conserve energy along its orbit

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v}) = -m\,\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}),$$

$$\boldsymbol{v}\cdot\frac{d}{dt}(m\boldsymbol{v}) + m\,\boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x}) = 0$$

But since $\displaystyle\frac{d\Phi}{dt} = \boldsymbol{v}\cdot\nabla\Phi(\boldsymbol{x})$

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\frac{d}{dt}\left[\frac{m}{2}(\boldsymbol{v}^2) + m\Phi(\boldsymbol{x})\right] = 0$$

where $\{\hat{x}, \hat{y}, \hat{z}\}$ are the unit vectors in their respective directions.

This is just the KE + PE

$$\frac{dL}{dt} = \boldsymbol{x}\times m\frac{d\boldsymbol{v}}{dt} = -m\,\boldsymbol{x}\times\nabla\Phi$$

Angular momentum L

Some Basics - M. Whittle

- The gravitational potential energy is a scalar field
- its gradient gives the net gravitational force (per unit mass) which is a vector field :

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla \cdot \nabla = \nabla^2$$

$$\Phi(\mathbf{r}) = -G \int_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} \, d^3\mathbf{r}'$$

$$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) = G \int_V \frac{\mathbf{r}' - \mathbf{r}}{|\mathbf{r}' - \mathbf{r}|^3} \, \rho(\mathbf{r}') \, d^3\mathbf{r}'$$

$$\nabla \cdot \mathbf{F}(\mathbf{r}) = -4\pi G \rho(\mathbf{r})$$

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G \rho(\mathbf{r})$$ ⟷ Poissons eq inside the mass distribution

$$\nabla^2\Phi(\mathbf{r}) = 0$$ ⟷ Outside the mass dist

Poisson's Eq+ Definition of Potential Energy (W)

So the force per unit mass is

$\rho(x)$ is the density dist

$$\boldsymbol{F}(\mathbf{x}) = -\nabla \Phi(\mathbf{x}) = \int G \rho(\mathbf{x}') \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}|^3} d^3\mathbf{x}'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides

S+G pg 112-113

$$\nabla^2 \Phi(\mathbf{x}) = -\nabla^2 \int \frac{G\rho(\mathbf{x}')}{|\mathbf{x} - x'|} d^3\mathbf{x}'$$

$$= 4\pi G \rho(\mathbf{x})$$ Poisson's equation.

Potential energy W

$$W = \frac{1}{2} \int_V \rho(\mathbf{r}) \, \Phi(\mathbf{r}) \, d^3\mathbf{r} = -\frac{1}{8\pi G} \int_V |\nabla\Phi|^2 \, d^3\mathbf{r}$$

Derivation of Poisson's Eq

So the force per unit mass is

$$\boldsymbol{F}(\boldsymbol{x})=-\nabla\Phi(\boldsymbol{x})=\int G\rho(\boldsymbol{x}')\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}|^3}d^3\boldsymbol{x}'$$

To get the differential form we start with the definiti
of Φ and applying ∇^2 to both sides

$$\nabla^2\Phi(\boldsymbol{x})=-\nabla^2\int\frac{G\rho(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3\boldsymbol{x}'$$

$$=4\pi G\rho(\boldsymbol{x})$$

Poisson's equation.

see S+G pg112 for detailed derivation or web page
'Poisson's equation'

Characteristic Velocities

$v^2_{circular}$=r dΦ(r)/dt=GM/r; v=sqrt(GM/r) Keplerian

velocity dispersion **σ^2=(1/ρ) $\int\rho$ ($\partial\Phi$(r,z)/∂z)dz**
or alternatively σ^2(R)=(4πG/3M(R) $\int$rρ(r) M(R) dr

escape speed =v_{esc}=sqrt(2Φ(r)) or Φ(r)=1/2v^2_{esc}
so choosing r is crucial

More Newton-Spherical Systems B&T 2.2

Newtons 1st theorem: a body inside a spherical shell has no net gravitational force from that shell; e.g. $\nabla\Phi(r)=0$

Newtons 2nd theorem: the gravitational force on a body outside a spherical shell is the same as if all the mass were at a point at the center of the shell.

Simple examples:
Point source of mass M; potential $\Phi(r)$ =-GM/r;
definition of circular speed; speed of a test particle on a circular orbit at radius r
$v^2_{circular}=r\, d\Phi(r)/dt$=GM/r; $v_{circular}$=sqrt(GM/r) ;Keplerian

velocity dispersion $\sigma^2=(1/\rho)\int\rho\,(\partial\Phi(r,z)/\partial z)dz$
escape speed =sqrt[$2\Phi(r)$]=sqrt(2GM/r) ; from equating kinetic energy to potential energy $1/2mv^2=|\Phi(r)|$

Escape Speed

- As r goes to infinity $\phi(r)$ goes to zero
- so to escape $v^2>2\phi(r)$; e.q. $v_{esc}=\text{sqrt}(-2\phi(r))$

- Alternate derivation using conservation of energy
- Kinetic+Gravitational Potential energy is constant
- $KE_1+U_1=KE_2+U_2$
- Grav potential =-GMm/r; KE=$1/2mv_{escape}^2$
- Since final velocity=0 (just escapes) and U at infinity=0
- $1/2mv_{escape}^2$-GMm/r=0

Gravity and Dynamics-Spherical Systems- Repeat

- Newtons 1st theorm : a body inside a a spherical shell has no net force from that shell $\nabla\phi$ =0
- Newtons 2nd theorm ; a body outside the shell experiences forces as if they all came from a point at the center of the shell-Gravitational force at a point outside a closed **sphere** is the same as if all the mass were at the center
  - This does not work for a thin disk- cannot ignore what is outside of a given radius
- One of the prime observables (especially for spirals) is the circular velocity; in general it is $V^2(R)/R=G(M<R)/R^2$ more accurate estimates need to know shape of potential
- so one can derive the mass of a flattened system from the rotation curve

----

- point source has a potential ϕ=-GM/r
- A body in orbit around this point mass has a circular speed v_c^2=r ϕd/dr=GM/r
- v_c=sqrt(GM/r); Keplerian
- Escape speed from this potential v_{escape}=sqrt(2ϕ)=sqrt(2GM/r) (conservation of energy KE=1/2mv^2_{escape})

Homogenous Sphere B&T sec 2.2.2

- Constant density sphere of radius a and density ρ_0
- $M(r)=4\pi Gr^3\rho_0$; r<a
- $M(r)=4\pi Ga^3\rho_0$; r>a

$\phi(R)=-d/dr(M(R))$; $\phi(R)=-3/5GM^2/R$; B&T 2.41)

R>a $\phi(r)=4\pi Ga^3\rho_0=-GM/r$

R<a $\phi(r)=-2\pi G\rho_0(a^2-1/3r^2)$);

$v^2_{circ}=(4\pi/3)G\rho_0 r^2$; solid body rotation R<a

Orbital period **$T=2\pi r/v_{circ}$=sqrt($3\pi/G\rho_0$)**

Dynamical time=crossing time =T/4=**sqrt($3\pi/16G\rho_0$)**

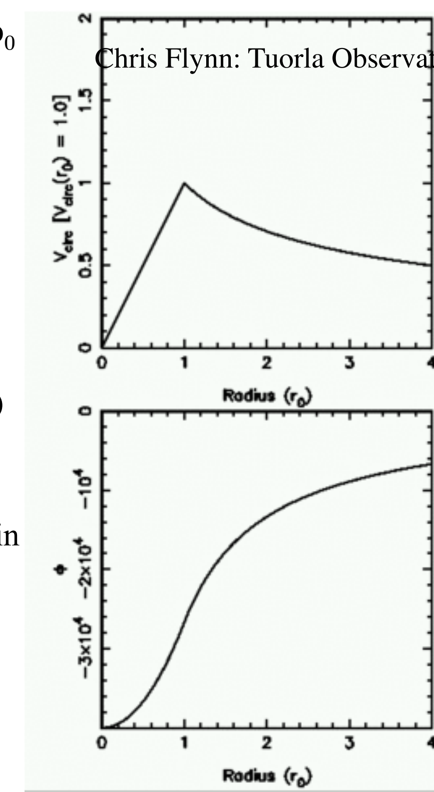
Potential is the same form as an harmonic oscillator with angular freq 2π/T (B&T 2.2.2(b))

Regardless of r a particle will reach r=0 (in free fall) in a time T=/4

Eq of motion of a test particle INSIDE the sphere is

$dr^2/dt^2=-GM(r)/r^2=-(4\pi/3)G\rho_0 r$

General result dynamical time ~sqrt(1/Gρ)



Some Simple Cases

- **Constant density sphere** of radius a and density ρ_0

Potential energy (B&T) eq 2.41, 2.32

$\phi(R)=-d/dr(M(R))$;

R>a $\phi(r)=4\pi Ga^3\rho_0=-GM/r$

R<a $\phi(r)=-2\pi G\rho_0(a^2-1/3r^2))$;

$v^2_{circ}= (4\pi/3)G\rho_0 r^2$ solid body rotation

Potential is the same form as a harmonic oscillator

e.g. the eq of motion is $d^2r/dt^2=-GM(r)/r=4\pi/3Gr\rho$; solution to harmonic oscillator is

$r=A\cos(\omega t+\phi)$ with $\omega=$ sqrt$(4\pi/3G\rho)=2\pi/T$

$T=$sqrt$(3\pi/G\rho_0)=2\pi r/v_{circ}$

Spherical Systems: Homogenous sphere of radius a Summary

- $M(r)=4/3\pi r^3\rho$ (r<a); r>a $M(r)=4/3\pi r^3 a$
- Inside body (r<a); $\phi(r)=-2\pi G\rho(a^2-1/3\ r^2)$ (from eq. 2.38 in B&T)

Outside (r>a);)$\phi(r)= -4\pi G\rho(a^3/3)$

Solid body rotation $v_c^2= -4\pi G\rho(r^2/3)$

Orbital period $T=2\pi r/v_c=$sqrt$(3\pi/G\rho)$;

a crossing time (dynamical time) $=T/4=$sqrt$(3\pi/16G\rho)$

potential energy $W=-3/5GM^2/a$

The motion of a test particle inside this sphere is that of a simple harmonic oscillator $d^2r/dt^2= -G(M(r)/\ r^2=4\pi G\rho r/3$ with angular freq $2\pi/T$

no matter the intial value of r, a particle will reach r=0 in the dynamical time T/4

In general the dynamical time $t_{dyn}\sim 1/$sqrt$(G<\rho>)$

and its 'gravitational radius' $r_g= GM^2/W$

Star Motions in a Simple Potential

- if the density ϱ in a spherical galaxy is constant, then a star following a circular orbit moves so that its angular speed $\Omega(r) = V(r)/r$ is constant.
- a star moving on a radial orbit, i.e., in a straight line through the center, would oscillate harmonically in radius with period
- $P = \text{sqrt}[3\pi/G\varrho] \sim 3t_{ff}$, where $t_{ff} = \text{sqrt}[1/G\varrho]$: S&G sec 3.1

Not so Simple - Plummer Potential sec 2.2 in B&T

- Many astrophysical systems have a 'core'; e.g. the surface brightness flattens in the center (globular clusters, elliptical galaxies, clusters of galaxies, bulges of spirals) so they have a characteristic length
- so imagine a potential of the form $-\phi(r) = -GM/\text{sqrt}(r^2+b^2)$; where b is the Plummer scale length

$\nabla^2\Phi(r) = (1/r^2)\, d/dr(r^2 d\phi/dr) = 3GMb^2/(r^2+b^2)^{5/2} = 4\pi G\rho(r)$ Poissons eq

and thus

$\rho(r) = (3GM/4\pi b^3)[1+(r/b)^2]^{-5/2}$

Now take limits $r \ll b$ $\rho(r) = (3GM/4\pi b^3)$ constant

$r \gg b$ $\rho(r) = (3GM/4\pi b^3) r^{-5}$ finite

Plummer potential was 'first' guess at modeling 'real' spherical systems; it is one of a more general form of 'polytropes' B&T (pg 300)

Potential energy $W = 3\pi GM^2/32b$

Spherical systems- Plummer potential

- Another potential with an analytic solution is the Plummer potential - in which the density is constant near the center and drops to zero at large radii - this has been used for globular clusters, elliptical galaxies and clusters of galaxies.

- One such form- Plummer potential
  ϕ=-GM/(sqrt(r^2+b^2); b is called a scale length

  The density law corresponding to this potential is
  (using the definition of $\nabla^2\phi$ in a spherical coordinates)

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right).$$

$\nabla^2\phi = (1/r^2)d/dr(r^2 d\phi/dr)=(3GMb^2)/((r^2+b^2)^2)^{5/2}$

$\rho(r)=(3M/4\pi b^3)(1+(r/b)^2)^{-5/2}$

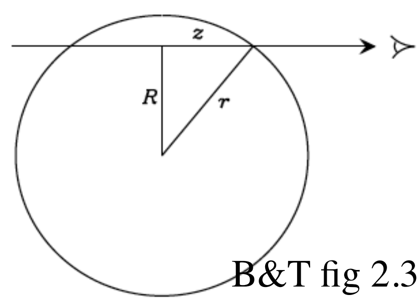
Potential energy $W=-3\pi GM^2/32b$

Many More Not So Simple Analytic Forms

- B&T pgs 65-72 ; there are many more forms which are better and better approximations to the true potential of 'spherical' systems; I will not cover them in detail in the lectures, please read the relevant sections of the text.
- However I will cover 2 others- the modified Hubble law which is frequently used for clusters of galaxies
- B&T eq 2.53 starts with the luminosity density $j=j_o(1+(r/a)^2)^{-3/2}$
- which gives surface brightness
  $I(r)=2aj_o(1+(r/a)^2)^{-1}$
- at r=a ; I(a)=1/2I(0); a is the core radius
- Now if light traces mass and the mass to light ratio is constant

$M=\int j(r)d^3r=$
$4\pi a^3 G j_o[\ln[R/a+\mathrm{sqrt}(1+(r/a)^2)]-(r/a)(1+(r/a)^{-1/2}\}$
B&T eq 2.56

- and the potential is also analytic



B&T fig 2.3

Problems: mass diverges logarithimically
BUT potential is finite and at r>>a is almost GM/r

Spherical Systems

- A frequently used analytic form for the surface brightness of an elliptical galaxy is the Modified Hubble profile

- $I(R)=2j(o)a/[(1+(r/a)^2]$ which has a luminosity density distribution
  $j(r)=j(0)[(1+(r/a)^2]^{-3/2}$
- this is also called the 'pseudo-isothermal' sphere distribution

- the eq for ϕ is analytic and finite at large r even though the mass diverges (eq. 2.56, 2.57 in B+T)
  $\phi=-GM/r-(4\pi G j_0 a)^2/\mathrm{sqrt}[1+(r/a)^2]$

Last Spherical Potential

- In the last 15 years numerical simulations have shown that the density distribution of dark matter can be well described by a form called 'NFW' density distribution (B+T eq 2.65)

$\rho(r)=\rho(0)/[(r/a)^{\alpha}(1+(r/a))^{\beta-\alpha}]$ with $(\alpha,\beta)= (1,3)$

Integrating to get the mass
$M(r)=4\pi G\rho(0)a^3\ln[1+(r/a)]-(r/a)/[1+(r/a)]$
and potential $\phi=[\ln(1+(r/a)]/(r/a)]$

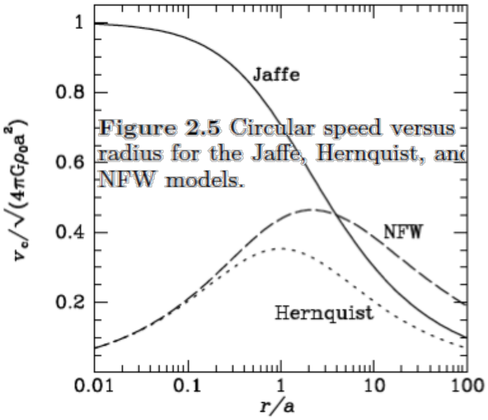
See problem 3.7 in S&G

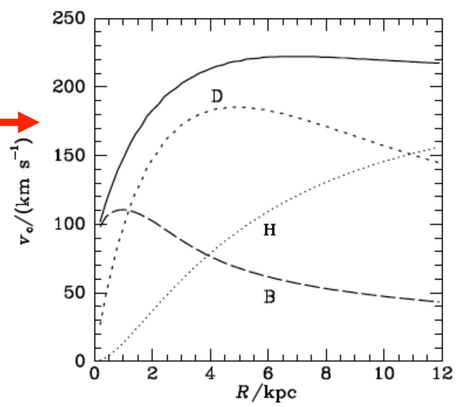
There is a long history of different potentials and B&T goes thru it... no longer relevant to modern work except to improve your skills !

The NFW density distribution is an analytic approximation to numerical simulations of cold dark matter

Other Forms

- B+T discuss many other forms which are interesting mathematically but are not really relevant to the rest of our class.
- However all the forms so far have a Keplerian rotation $v \sim r^{-1/2}$ while real galaxies have flat rotation curves $v_c(R)=v_0$
- A potential with this property must have $d\phi/dr=v_0^2/R$; $\phi=v_0^2\ln R+C$
- However this is a rather artificial form; real galaxies seem to be composed of 3 parts: disk (D), bulge (B), halo (H) and it is the sum of the 3 that gives the flat rotation curve (very fine tuned and very flexible)





Summary of Dynamical Equations

- **gravitational pot'l $\Phi(r)=-G\int\rho(r)/|r-r'|\, d^3r$**
- **Gravitational force $F(r)= -\nabla\Phi(r)$**
- **Poissons Eq $\nabla^2\Phi(r)= 4\pi G\rho$; if there are no sources**

  Laplace Eq $\nabla^2\Phi(r)= 0$
- **Gauss's theorem** : $\int\nabla\Phi(r)\bullet ds^2=4\pi GM$
- **Potential energy $W=1/2\int r\rho(r)\nabla\Phi d^3r$**

- In words Gauss's theorem says that the integral of the normal component of $\nabla\Phi$ over and closed surface equals $4\pi G$ times the mass enclosed

Potentials are Separable

- We make the fundamental assumption that the potential of a system can be decomposed into separable parts-
- This is because Poisson's equation is linear :
- differences between any two ϕ—ρ pairs is also a ϕ—ρ pair, and

differentials of ϕ—ρ or are also ϕ—ρ pairs

- e.g. $\phi_{total}=\phi_{bulge}+\phi_{disk}+\phi_{halo}$

So Far Spherical Systems

- But spiral galaxies have a significant fraction of the mass (?; at least the baryons) in a flattened system.

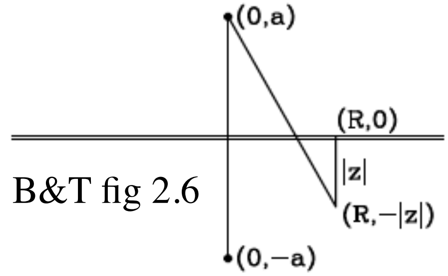
Kuzmin Disk B&T sec 2.3 S&G Prob 3.4;

- This ansatz is for a flattened system and separates out the radial and z directions
- Assume $\phi_K(z,R)= GM/[\mathrm{sqrt}(R^2+(a+z)^2)]$; axisymmetric (cylindrical)

  R is in the x,y plane
- Analytically, outside the plane, ϕ_K has the form of the potential of a point mass displaced by a distance 'a' along the z axis
  - e.q. R(z)= $\begin{cases} (0, a); z<0 \\ (0, -a); z>0 \end{cases}$
- Thus $\nabla^2\Phi=0$ everywhere except along z=0- Poisson's eq
- Applying Gauss's thm $\int\nabla\Phi d^2s=4\pi GM$

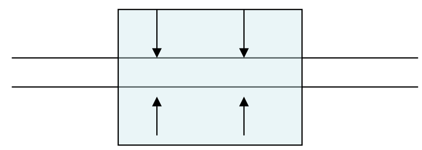
and get $\Sigma(R)=aM/[2\pi(R^2+a^2)^{3/2}]$

this is in infinitely thin disk... not too bad an approx



B&T fig 2.6

Use of Gauss's thm (divergence)
the sum of all sources minus the sum of all sinks gives the net flow out of a region.



$\int\nabla\Phi d^2s=4\pi GM=2\pi G\Sigma$

as $z \longrightarrow 0$; $\mathbf{\Sigma=(1/2\pi)G\, d\Phi/dr}$

Isothermal Sheet MBW pg 498

- simple model for the vertical structure of disk galaxies
- Allows an estimate the disk mass from a measurement of the vertical velocity dispersion, σ_z, and the radial scale length, R_d, if one knows the vertical scale height of the tracer population
- The relevant Poisson eq is $d^2\phi_z/d\,(z/z_d)^2=1/2\exp-(\phi_z)$;
- $\phi_z=\phi/\sigma^2_z$ *and* $z_d = \sigma_z/\mathit{sqrt}(8\pi G\rho(R,0))$

$\sigma^2_z(R) =(z/z_d)GM_dR_d\exp(-RR_d)$

- where z_d is the vertical scale height of the disk and R_d is the radial scale length
- can solve for the density distribution the disk

- *Why do we want to do this??- Estimates of the mass for face on galaxies where radial velocity data are impossible.*

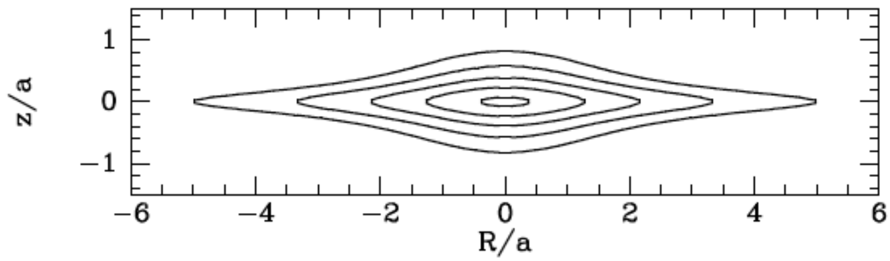
Flattened +Spherical Systems-B&T eqs

- Add the Kuzmin to the Plummer potential
- When b/a~ 0. 2, qualitatively similar to the light distributions of disk galaxies,

$$\Phi_M(R,z) = -\frac{GM}{\sqrt{R^2 + (a + \sqrt{z^2 + b^2})^2}}. \tag{2.69a}$$

When $a = 0$, Φ_M reduces to Plummer's spherical potential (2.44a), and when $b = 0$, Φ_M reduces to Kuzmin's potential of a razor-thin disk (2.68a). Thus, depending on the choice of the two parameters a and b, Φ_M can represent the potential of anything from an infinitesimally thin disk to a spherical system. If we calculate $\nabla^2\Phi_M$, we find that the mass distribution with which it is associated is (Miyamoto & Nagai 1975)

$$\rho_M(R,z) = \left(\frac{b^2 M}{4\pi}\right) \frac{aR^2 + (a + 3\sqrt{z^2+b^2})(a + \sqrt{z^2+b^2})^2}{\left[R^2 + (a + \sqrt{z^2+b^2})^2\right]^{5/2}(z^2+b^2)^{3/2}}. \tag{2.69b}$$



Contours of equal density in the (R; z) plane for b/a=0.2

Potential of an Exponential Disk B&T sec 2.6

- As discussed earlier the light profile of the stars in most spirals has an exponential scale LENGTH

$\Sigma(R)=\Sigma_0\exp(-R/R_d)$ (this is surface brightness NOT surface mass density)- see next page for formula's

Do we learn anything from this ?? see MWB 11.1.2

Fig 2.17 in B&T - how the circular speed (a potential observable) depends on the scale length for different mass distributions.

> Mass of exponential disk
> $M(R)= \int\Sigma(R)Rdr =$
> $2\pi\Sigma_0 R_d^2[1-\exp(R/R_d)(1+R/R_d)]$
>
> when R gets large
> $M\sim 2\pi\Sigma_0 R_d^2$

Potential of an Exponential Disk B&T sec 2.6

.

The circular velocity peaks at $R \sim 2.16\, R_d$ approaches Keplerian for a point mass at large R (eq. 11.30 in MWB) and depends only on Σ_0 and R_d

As long as the vertical scale length is much less than the radial scale the vertical distribution has a small effect - e.g. separable effects !

IF the disk is made only of stars (no DM) and and if they all have the same mass to light ratio Γ , R_d is the scale length of the stars , then the observables $I_0, R_d, v_{circ}(r)$ have all the info to calculate the mass!

Mass of exponential disk
$M(R) = \int \Sigma(R) R dr = 2\pi\Sigma_0 R_d^2[1-\exp(R/R_d)(1+R/R_d)]$

when R gets large
$M \sim 2\pi\Sigma_0 R_d^2$

Explaining Disks

- Remember the most important properties of disk dominated galaxies (MBW pg 495)
  - Brighter disks are on average
    - larger, redder, rotate faster, smaller gas fraction
  - flat rotation curves
  - surface brightness profiles close to exponential
  - lower metallicity in outer regions
  - traditional to model them as an infinitely thin exponential disk with a surface density distribution $\Sigma(R)=\Sigma_{0y}\exp(-R/R_d)$
  - This gives a potential (MBW pg 496) which is a bit messy

$\phi(R, z)=-2\pi G\Sigma_0{}^2 R_D \int [J_0(kR)\exp(-k|z|)]/[1+(kR_D)^2]^{3/2} dk$

Exponential Disks

- Motivated by the exponential surface brightness profiles of disks examine a potential that is generated by such a distribution (B+T 2.162)

  $\Sigma(R)=\Sigma_0 \exp(-R/R_D)$ which gives a mass distribution

  $M(R)=2\pi\int\Sigma(R)RdR = 2\pi\Sigma_0 R^2{}_D[1-\exp(-R/R_D)(1+R/R_D)]$;

  as shown in detail in eqs B&T 2.153-2.157 one gets a potential in the form of Bessel functions

  This comes from the use of Hankel functions (analogs of Fourier transforms but for cylindrically symmetric systems)

  $S(k)=-2\pi G \int J_0(kR)\Sigma(R)RdR$; J_0 is a Bessel function order zero

  $\phi(R, z)=-2\pi G\Sigma_0{}^2 R_D\int[J_0(kR)\exp(-k|z|)]/[1+(kR_D)^2]^{3/2}dk$