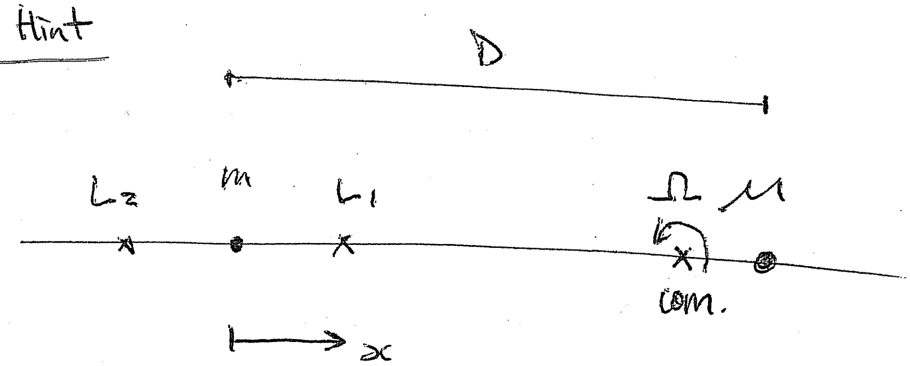
S&G 4.5 Hint



The Lagrange points $L_1 - L_3$ are local maxima of the effective potential $\Rightarrow$ a test particle at one of the points experiences no net force along x.

($\underline{F} = -\nabla \Phi_{eff}$, hence eq. (4.7) solving for $\frac{\partial \Phi_{eff}}{\partial x} = 0$)

Step 1: Write down force balance:

$$0 = \underbrace{\mp \frac{Gm}{x^2}}_{\text{attraction to } m} + \underbrace{\frac{GM(D-x)}{(D-x^2)}}_{\text{attraction to halo mass}} - \Omega^2 \Big(\overbrace{\frac{DM}{M+m} - x}^{\text{distance to c.o.m from } x} \Big)$$

similar to Eq. 4.7 but signs labelled differently (I think this is clearer!)

attraction to m (−ve if $x > 0$, +ve if $x < 0$)

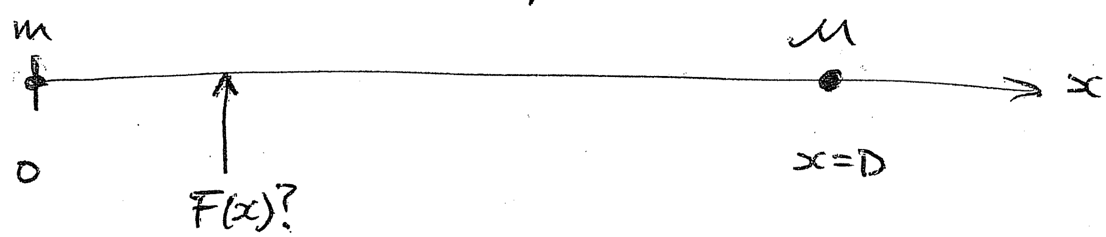
attraction to halo mass, always +ve; since halo is extended, the effective mass felt depends on x.

centrifugal force, −ve (away from c.o.m) M and Ω here are from the two-body problem, so $M = M(<D)$ and $\Omega^2 = \frac{G(M+m)}{D^3}$ (Kepler-3) Both are independent of x.

This is the term that is different for the halo mass, compared to a point mass, since $M(< D-x) \downarrow$ as we move away from the halo and vice versa.

Step 2 : understand what's different

Question hints that you should replace the "force from the dark halo mass" — this explains the difference.



$\boxed{\text{Case A}}$ M is a point mass. Then the force as a function of distance r from M is just

$$F = \frac{GM}{r^2}. \qquad (*)$$

$$\Rightarrow F(x) = F(0) + x \left.\frac{dF}{dx}\right|_{x=0} + \cdots \qquad \text{Taylor expansion.}$$

$$= \frac{GM}{D^2} + x\left[+\frac{2GM}{D^3}\right] + \cdots$$

(since dx = - dr)

$\boxed{\text{Case B}}$ M is now an extended halo described by

$$M(<r) = \frac{r v_H^2}{G}\left(1 - \frac{a_H}{r}\arctan\left(\frac{r}{a_H}\right)\right)$$

And the force is, at distance r,

$$F(r) = \frac{v_H^2}{r}\left(1 - \frac{a_H}{r}\arctan\left(\frac{r}{a_H}\right)\right) \approx \frac{v_H^2}{r}, \quad r \gg a_H \qquad (\dagger)$$

$F(r) = \frac{GM(<r)}{r^2}.$ where v_H^2 can be related to $M = M(<D)$ by

$$v_H^2 \approx \frac{GM}{D}$$

By inspection of the forms of $(*)$ and $(\dagger)$, one is $\propto \frac{1}{r^2}$ while the other is $\propto \frac{1}{r}$, so the perturbed force has a different $\left.\frac{dF}{dx}\right|_{x=0}$ change. This leads eventually to a different fraction in the solution for $\left(\frac{x}{D}\right)$.