Astronomy 540 – Homework 2

Due Wednesday, October 18, 2006

1. (B&T 3-4) Prove that if a homogeneous sphere of pressureless fluid with density ρ is released from rest, it will collapse to a point in a time $t_{ff} = \frac{1}{4}\sqrt{3\pi/(2G\rho)}$. The time t_{ff} is called the free-fall time of a system of density ρ.

2. (B&T 4-25) We may study the vertical structure of a thin axisymmetric disk by neglecting all the radial derivatives and adopting the form $f = f(E_z)$ for the DF, where $E_z \equiv \frac{1}{2}v_z^2 + \Phi(z)$.
(a) Using the thin disk approximation for Poisson's equation given by

$$\frac{\partial^2 \Phi(R,z)}{\partial^2 z} = 4\pi G\rho(R,z), \tag{1}$$

show that if $f = \rho_0(2\pi\sigma_z^2)^{-1/2} exp(-E_z/\sigma_z^2)$, Poisson's equation may be written

$$2\frac{\partial^2 \phi}{\partial^2 \zeta} = e^{-\phi}, \tag{2}$$

where $\phi \equiv \Phi/\sigma_z^2$, $\zeta \equiv z/z_0$, and $z_0 \equiv \frac{\sigma_z}{\sqrt{8\pi G\rho_0}}$.
(b) By solving this equation subject to the boundary conditions $\phi(0) = d\phi/d\zeta|_0 = 0$, show that the density ρ in the disk is given by (Spitzer 1942)

$$\rho(z) = \rho_0 \mathrm{sech}^2(\frac{1}{2}z/z_0). \tag{3}$$

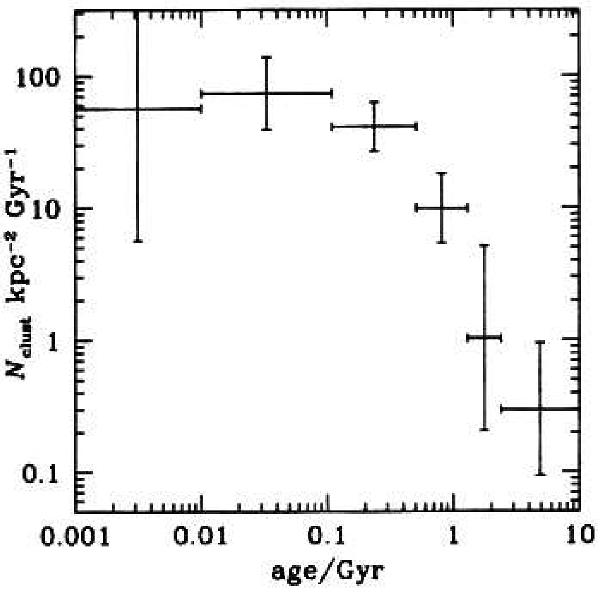
3. (B&T 8-11) A black hole of mass M is embedded in the center of an infinite, homogeneous, three-dimensional sea of test aprticles. Far from the hole, the test particles have a Maxwellian velocity distribution,

$$f_0(\mathbf{v}) = \frac{\nu_0}{(2\pi\sigma^2)^{3/2}} e^{-\frac{1}{2}v^2/\sigma^2}. \tag{4}$$

Show that the density distribution of test particles that are not bound to the hole is

$$\nu(r)/\nu_0 = 2\sqrt{r_H/\pi r} + e^{r_H/r}[1 - \mathrm{erf}(\sqrt{r_H/r})], \tag{5}$$

where $r_H \equiv GM/\sigma^2$. Show that close to the hole ($r \ll r_H$), $\nu(r) \propto r^{-1/2}$. Thus there is a weak density cusp around the hole.



4. Assume that the drop in the number of open clusters shown in the figure above arises from clusters dispersing into the field. Assume also that each cluster initially contains 1000 stars. Estimate the total number of stars per square parsec that these clusters have contributed to the field. If these stars end up distributed in a layer 100pc thick (comparable to the typical distances of open clusters from the galactic plane), estimate the number density of sush stars in the solar neighborhood. What fraction of nearby stars were born in such clusters? Why might this calculation significantly under-represent the total contributions made by clusters and associations to the field? Indicate what assumptions you have made in your calculations.