Galaxy Dynamics part 2

Physics of Galaxies 2013

part 10

university of groningen / faculty of mathematics and natural sciences / astronomy

Orbits of stars in spherical systems

- In a *time-independent* gravitational potential, **energy is conserved**:

$$\mathcal{E} \equiv \frac{1}{2}m\mathbf{v}^2 + m\Phi(\mathbf{x}) = \text{constant}$$

  - because $\partial\Phi/\partial t = 0$

- In a *spherical* potential, **the angular momentum is also conserved**: $\mathcal{L} \equiv \mathbf{x} \times m\mathbf{v}$
  - so $$\frac{d\mathcal{L}}{dt} = \mathbf{x} \times \frac{m\,d\mathbf{v}}{dt} = -m\mathbf{x} \times \nabla\Phi$$
  - but the force $\nabla\Phi$ always points towards the center, so $\mathcal{L}$ does not change

- Therefore the motion of a star is restricted to an *orbital plane*
  - and only two coordinates are needed to describe the location of the star
    - typically, polar coordinates in the plane (r, ϕ) are used to describe the motion

Orbits in an axisymmetric galaxy

- In an axisymmetric system, like a disk, we use (of course) a cylindrical coordinate system (R, ϕ, z) where z=0 corresponds to the symmetry plane (e.g., the mid-plane of the disk)
  - In an axisymmetric system, the mass distribution and therefore the potential is independent of the angular coordinate: $\partial\Phi/\partial\phi = 0$
    - no force in the ϕ-direction: **stars conserve angular momentum about the *z*-axis**
      - neglect non-axisymmetric features like the bar and the spiral arms!

- The equations of motion for a star in the disk are

$$\frac{d^2\mathbf{r}}{dt^2} = -\nabla\Phi$$

- In each direction, using $\mathbf{r} = R\hat{\mathbf{R}} + z\hat{\mathbf{z}}$, we have

$$\frac{d^2R}{dt^2} - R\left(\frac{d\phi}{dt}\right)^2 = -\frac{\partial\Phi}{\partial R} \quad (1)$$

$$\frac{d^2z}{dt^2} = -\frac{\partial\Phi}{\partial z} \quad (2)$$

$$\frac{d(R^2 d\phi/dt)}{dt} = -\frac{\partial\Phi}{\partial\phi} = 0 \quad (3)$$

remember, no force in the ϕ-direction!

- Equation (3) implies conservation of angular momentum around the *z*-axis:

$$L_z \equiv R^2\frac{d\phi}{dt} = \text{constant}$$

- In the R-direction, we can rewrite Equation (1) as

$$\frac{d^2R}{dt^2} = R\left(\frac{d\phi}{dt}\right)^2 - \frac{\partial\Phi}{\partial R} = -\frac{\partial\Phi_{\rm eff}}{\partial R} \tag{4}$$

    - where the *effective potential* $\Phi_{\rm eff}(R, z; L_z)$ is

$$\Phi_{\rm eff} \equiv \Phi(R, z) + \frac{L_z^2}{2R^2}$$

- If we multiply Equation (4) by dR/dt and integrate with respect to t, we find

$$\frac{1}{2}\left(\frac{dR}{dt}\right)^2 + \Phi_{\rm eff}(R, z; L_z) = \text{constant}$$

    - a sort of energy conservation law!

- Therefore the effective potential $\Phi_{\text{eff}}(R, z; L_z)$ acts as a *potential energy* for the star's motion in *R* and *z*

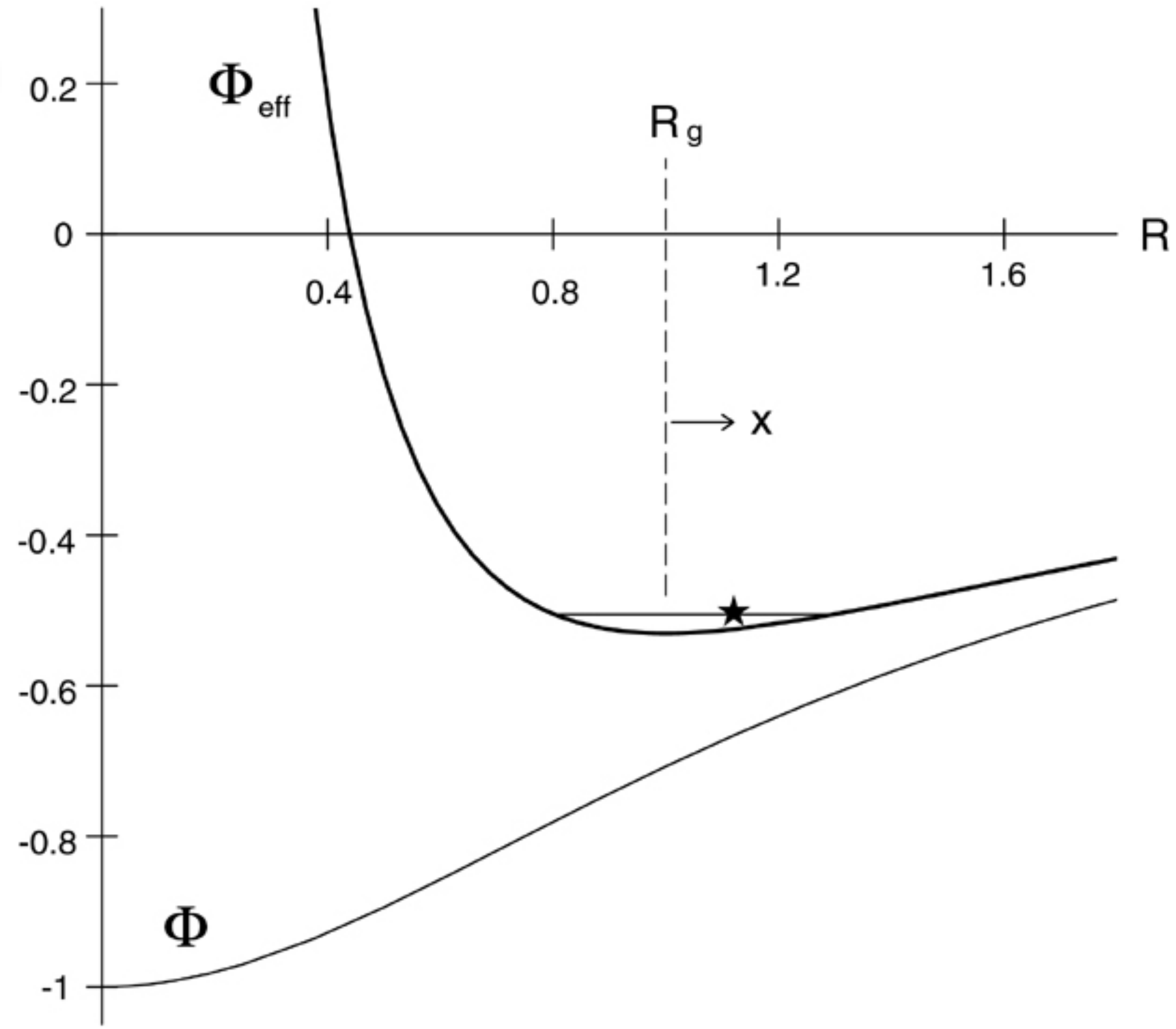
- The effective potential is constant if
  - $\dfrac{\partial \Phi_{\text{eff}}}{\partial R} = 0 \text{ and thus } \dfrac{\partial \Phi}{\partial R} - \dfrac{L_z^2}{R^3} = 0$
  - and $\dfrac{\partial \Phi_{\text{eff}}}{\partial z} = \dfrac{\partial \Phi}{\partial z} = 0$

- The second equation is satisfied for motion in the mid-plane, so that *z*=0; combined with $dR/dt = 0$, this implies a *circular orbit* in the disk mid-plane
  - The radius of this circular orbit is R_g, where

$$\left.\frac{\partial \Phi}{dR}\right|_{R_g} = \frac{L_z^2}{R_g^3} = R_g \left(\frac{d\phi}{dt}\right)^2$$

remember the definition of L_z!

- Since $\dot{R}^2 \geq 0$, the L_z^2 term in the effective potential acts as an *angular momentum barrier* preventing a star coming closer to *R*=0 than some perigalactic radius where $\dot{R} = 0$
- The circular orbit is that with the least energy for a given angular momentum L_z



Effective potential for a star with L_z=0.595 in a Plummer potential

Epicycles

- Let's now derive approximate solutions for the equations of motion for stars on (nearly) circular orbits in the disk (symmetry) plane
- Define a distance around R_g, $x = R - R_g$, and expand the effective potential around (R_g,0):

$$\Phi_{\rm eff}(R,z) \sim \Phi_{\rm eff}(R_g,0) + \left.\frac{\partial \Phi_{\rm eff}}{\partial R}\right|_{R_g,0} x + \left.\frac{\partial \Phi_{\rm eff}}{\partial z}\right|_{R_g,0} z + \cdots$$

- Let's define two quantities:

$$\kappa^2 = \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right|_{R_g,0} \quad \text{and} \quad \nu^2 = \left.\frac{\partial^2 \Phi_{\rm eff}}{\partial z^2}\right|_{R_g,0}$$

- Then the equations of motion become

$$\frac{d^2R}{dt^2} = -\frac{\partial \Phi_{\rm eff}}{\partial R} \text{ so } \frac{d^2x}{dt^2} = -x\left.\frac{\partial^2 \Phi_{\rm eff}}{\partial R^2}\right|_{R_g,0} \implies \boxed{\frac{d^2x}{dt^2} = -\kappa^2 x}$$

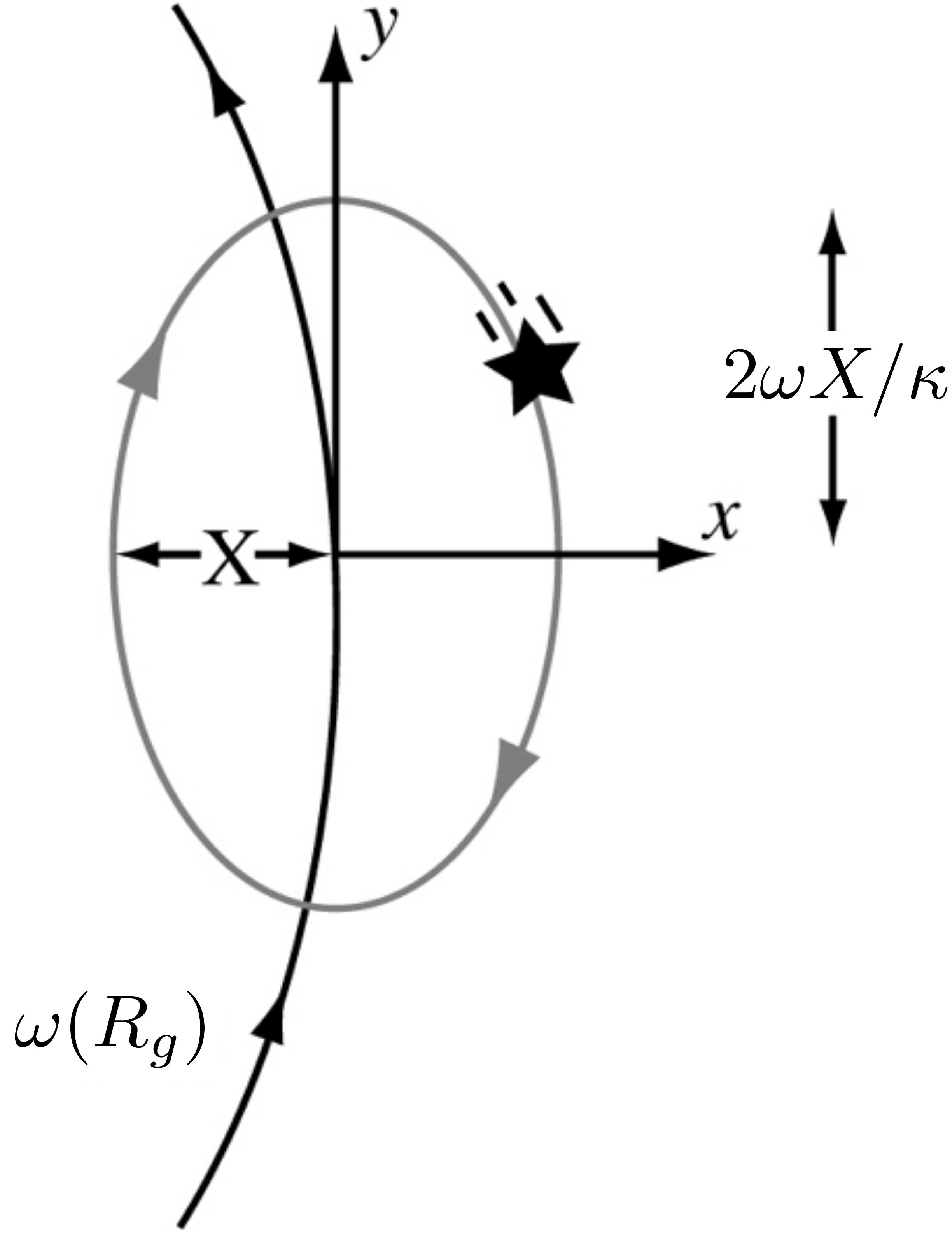
$$\frac{d^2z}{dt^2} = -\frac{\partial \Phi_{\rm eff}}{\partial z} \text{ so } \frac{d^2z}{dt^2} = -z\left.\frac{\partial^2 \Phi_{\rm eff}}{\partial z^2}\right|_{R_g,0} \implies \boxed{\frac{d^2z}{dt^2} = -\nu^2 z}$$

- These are *the equations of motion of two decoupled harmonic oscillators* with frequencies κ and ν
  - κ is the **epicyclic frequency**: $\kappa^2(R_g) = \left.\frac{\partial^2 \Phi}{\partial R^2}\right|_{R_g,0} + 3\frac{L_z^2}{R_g^4}$
  - ν is the **vertical frequency**: $\nu^2 = \left.\frac{\partial^2 \Phi}{\partial z^2}\right|_{R_g,0}$

- The solution to the equations of motion is then

$$x = X_0 \cos(\kappa t + \Psi) \text{ for } \kappa^2 > 0$$
$$z = Z_0 \cos(\nu t + \theta) \text{ for } \nu^2 > 0$$

- The motion of a star in the disk can be described as the oscillation about a *guiding center* moving on a circular orbit at R_g



- In order to conserve angular momentum L_z, the azimuthal speed must also vary:

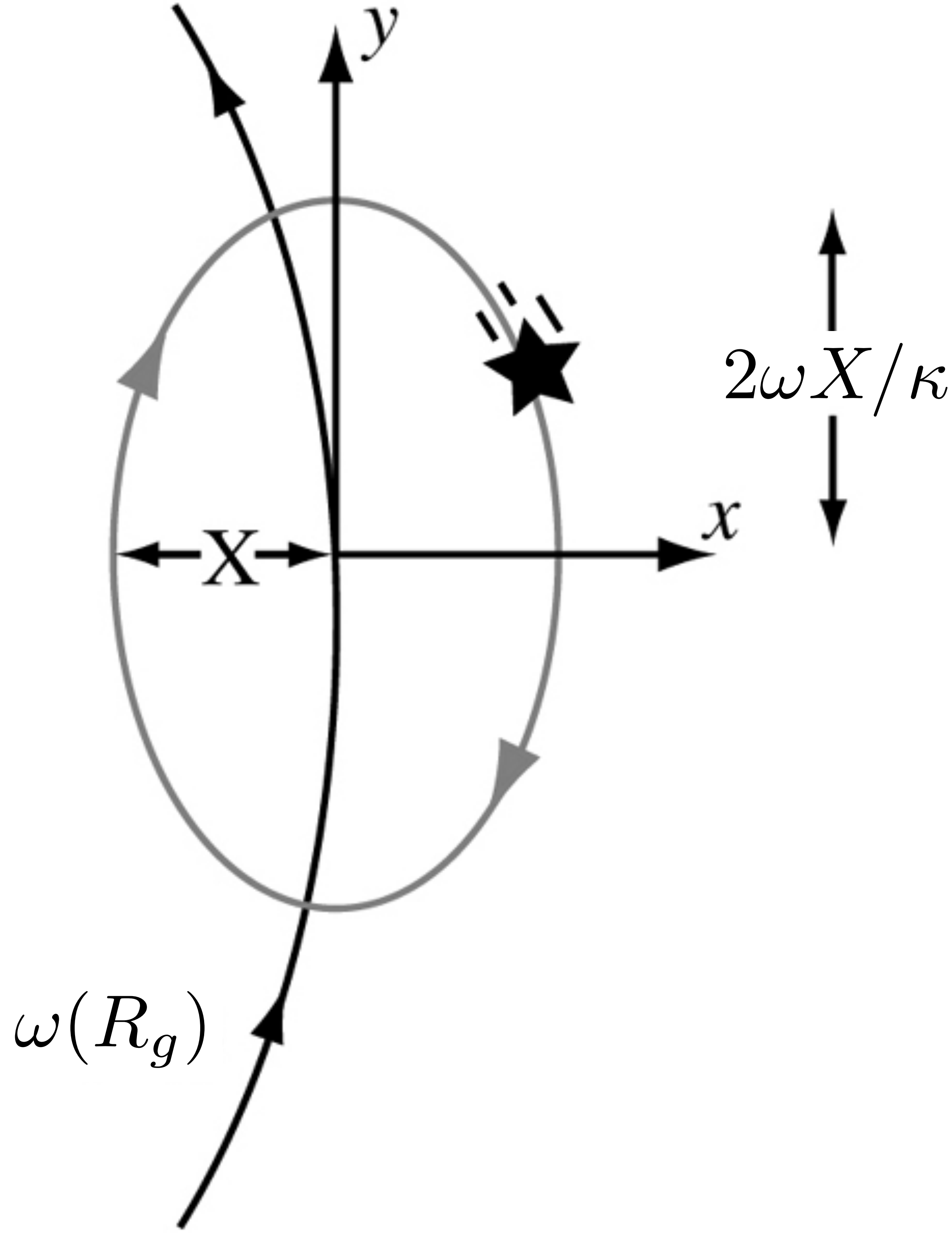
$$\frac{d\phi}{dt} = \frac{L_z}{R^2} = \frac{\omega(R_g)R^2}{(R_g + x)^2} \approx \omega(R_g)\left(1 - \frac{2x}{R_g} + \cdots\right)$$

- Integrating, we find

$$\phi(t) = \phi_0(t) + \omega_g t - \frac{1}{R_g}\frac{2\omega_g}{\kappa}X_0 \sin(\kappa t + \Psi)$$

  - The first two terms give the guiding center motion (ϕ_0 is an arbitrary constant); the third term is harmonic motion with the same frequency as the *x* oscillation but 90° out of phase and with a different amplitude

- This motion is known as the **epicyclic motion**
  - It is retrograde because it is in the opposite sense to the guiding center's motion: speeds the star up closer to the center and slows it down farther out



- Note that the approximation to second order in *z* in the effective potential $(\Phi_{\rm eff} \propto z^2)$ is only valid if the density is constant in the *z*-direction (since $\nabla^2\Phi \sim \rho$). But the disk density decreases exponentially away from the mid-plane — so the approximation is valid **at most** one scale height away from the plane (*z*<300 pc). Since a good fraction of the disk stars move to greater heights, the motion in the *z*-direction is not well-described as a harmonic oscillation...

- There is a relation between the epicyclic frequency **κ** and the angular frequency **ω**:
  - First, the centrifugal force=gravitation pull, so

$$R\omega^2 = \frac{\partial \Phi}{\partial R}$$

  - and $\omega^2 = \frac{L_z^2}{R^4}$
  - so $\kappa^2 = \left(R \frac{d\omega^2}{dR} + 4\omega^2 \right)_{R_g}$
- In general, $\omega \leq \kappa \leq 2\omega$
  - for a sphere of uniform density, $\omega(R) = \text{cnst and } \kappa = 2\omega$
  - for the Kepler problem (a point mass), $\omega \propto r^{-3/2} \text{and } \kappa = \omega$

- So the epicyclic frequency is also related to the Oort constants...
  - Recall that $A = -\frac{1}{2} R \left. \frac{d\omega}{dR} \right|_{R_0}$ and $B = -\left. \left(\omega + \frac{1}{2} R \frac{d\omega}{dR} \right) \right|_{R_0}$
  - so at the Sun, $\kappa_0^2 = -4B(A - B) = -4B\omega_0$
  - Using the measured value of *B*, $\kappa_0 / \omega_0 \sim 1.3 \pm 0.2$
    - So the Sun makes 1.3 radial oscillations in the time it takes to make one complete revolution around the Galactic Center...

The Collisionless Boltzmann Equation (CBE)

- All of these equations can actually be derived from the "collisionless Boltzmann equation" (or the *Liouville equation*), which describes the paths of stars in **phase space**
  - Let's assume we have a large number of stars moving in a *smooth* potential
  - Then phase space is well-populated and a **distribution function** applies: $f(\mathbf{x}, \mathbf{v}, t) d^3\mathbf{x}\, d^3\mathbf{v}$ is the number of particles per unit volume in phase space $(\mathbf{x}, \mathbf{v})$

- Since the potential is smooth, neighboring particles in phase space (at the same **x**,**v**) move together, and we can use the *fluid approximation*:

$$\mathbf{w} = (\mathbf{x}, \mathbf{v})$$
$$\dot{\mathbf{w}} = (\mathbf{v}, -\nabla\Phi)$$

- Finally, the flow is **smooth**: stars do *not* jump discontinuously from one region of phase space to another — **no collisions**
  - Then *f* has to satisfy a **continuity equation**:

$$\frac{\partial f}{\partial t} + \nabla_6 \cdot (f\dot{\mathbf{w}}) = 0 \qquad (5)$$

phase space divergence

phase space "current"

- Expand equation (5):

$$\frac{\partial f}{\partial t} + f\nabla_6 \cdot (\dot{\mathbf{w}}) + \dot{\mathbf{w}} \cdot \nabla_6 f = 0$$

- Consider the term $\nabla_6 \cdot (\dot{\mathbf{w}})$. This is the divergence of the flow in phase space. Let's now show that this is *incompressible*:

$$\nabla_6 \cdot (\dot{\mathbf{w}}) = \sum_{i=1}^{6} \frac{\partial \dot{w}_i}{\partial w_i} = \sum_{i=1}^{3} \left(\frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right) = 0$$

v_i and x_i are **independent**

assumed a potential, so $\partial \dot{v}_i / \partial v_i = 0$

- Flow in phase space is incompressible *as long as* the forces do not depend on particle velocity

- The result is the CBE for incompressible flow in phase space:

$$\frac{\partial f}{\partial t} + \dot{\mathbf{w}} \cdot \nabla_6 f = 0$$

- or (6)

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0$$

- This means that the **convective derivative**, the change in the local density of phase-space particles seen by an observer moving with the phase-space fluid at a velocity $\dot{\mathbf{w}}$, is zero:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f}{\partial w_i} = 0$$

Note that this is also called the "Lagrangian derivative"

- This **Liouville's theorem**: *phase-space densities along particle trajectories are **constant***
    - Two key assumptions:
        - No small scale structure in the potential inside a volume element and no discontinuous jumps out of the volume element, so stars in the element are conserved: **collisionless system**
        - No frictional drag or encounters where energy and/or momentum can be exchanged with other stars: **no entropy increase**

Moments of the CBE: the Jeans Equations

- We are going to “take moments” of the CBE by multiplying *f* by powers of *v*
- First, let’s define the space density (of the stellar population component we’re interested in) as

$$\nu \equiv \int f \, d^3\mathbf{v}$$

  - This is the **zeroth moment** in the velocity of the distribution function

- The **first** and **second** moments in velocity of *f* are

$$\langle v_i \rangle = \frac{1}{\nu} \int v_i f \, d^3\mathbf{v}$$

$$\langle v_i v_j \rangle = \frac{1}{\nu} \int v_i v_j f \, d^3\mathbf{v}$$

- Now, the **zeroth moment** (in velocity) **of the CBE** is

$$\int \frac{\partial f}{\partial t} d^3\mathbf{v} + \int v_i \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0$$

  - note that here I've used the *summation convention*, where repeated indices are summed over

- This can be rewritten as

$$\frac{\partial}{\partial t}\int f\, d^3\mathbf{v} + \frac{\partial}{\partial x_i}\int v_i f\, d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i}\int\int dv_1 dv_2 \int_{v_3=-\infty}^{v_3=+\infty} \partial f = 0$$

  - But the last term is 0, because

$$f(v_i)|_{-\infty}^{+\infty} = 0 \text{ if } \lim_{v_i \to \infty} f = 0$$

- And so $$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}(\nu v_i) = 0$$
- or $$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \mathbf{v}) = 0 \qquad (7)$$
  - which is the *continuity equation* in **x** space (conservation of mass)

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- The **first moment** in velocity **of the CBE** is

$$\int v_j \frac{\partial f}{\partial t} d^3\mathbf{v} + \int v_i v_j \frac{\partial f}{\partial x_i} d^3\mathbf{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3\mathbf{v} = 0$$

- Now,

$$\begin{aligned} \int v_j \frac{\partial f}{\partial v_i} d^3\mathbf{v} &= \int\int dv_1 dv_2 \int_{v_3=-\infty}^{v_3=+\infty} \partial f - \int \left(\frac{\partial v_j}{\partial v_i}\right) f \, d^3\mathbf{v} \\ &= 0 - \delta_{ij}\nu \end{aligned}$$

- so $\dfrac{\partial}{\partial t}(\nu\langle v_j\rangle) + \dfrac{\partial}{\partial x_i}(\nu\langle v_i v_j\rangle) + \nu\dfrac{\partial \Phi}{\partial x_j} = 0$

- Subtracting off the zeroth moment, we have

$$\nu \frac{\partial \langle v_j \rangle}{\partial t} - \langle v_i \rangle \frac{\partial}{\partial x_i}(\nu \langle v_j \rangle) + \frac{\partial}{\partial x_i}(\nu \langle v_i v_j \rangle) + \nu \frac{\partial \Phi}{\partial x_j} = 0$$

- We usually rewrite this in terms of the *velocity dispersion tensor*, the velocity dispersion about the mean (streaming) motion:

$$\sigma_{ij}^2 = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle$$

- Substituting and rearranging, we finally have

$$\nu \frac{\partial \langle v_j \rangle}{\partial t} + \nu \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i} \tag{8}$$

  - The left-hand side of this equation is the Lagrangian derivative of the **momentum:** $\nu D\mathbf{v}/Dt$
  - The right-hand side is the sum of the **gravity** and a **stress term**, which is the *anisotropic pressure*
  - This equation is a **force equation**: the conservation of momentum along particle trajectories

- Since the velocity dispersion tensor σ_{ij} is symmetric, it can be *diagonalized*. This diagonalized tensor is the **velocity ellipsoid**, which we met earlier...

- So what use are the Jeans equations (moments of the CBE)?
  - Can relate observables like ν, **v**, and σ_{ij}^2 to the gravitational potential $\partial\Phi/\partial x_i$
    - they give us a way to “weigh” galaxies

- But!
  - Jeans equations describe a *massless* *tracer population* in an external potential — need to add Poisson's equation to get the potential from the (total) mass density **ρ**: Jeans equations are **incomplete**
  - There is no "equation of state" relating **v** to **σ** (like the ideal gas law relates **ρ** and T in a gas). You have to assume σ_{ij}, and thus *f*(**v**) — every different assumption leads to a different solution: Jeans equations depend on *f*(**v**) and are *non-unique*
  - The equations **never close**; have always to assume a higher-order tensor, i.e., some form for *f*(**v**).

Asymmetric drift: an application of the Jeans eqn

- Let's evaluate the quantity $v_a \equiv v_c - \langle v_\phi \rangle$
  - the amount a population lags behind the circular velocity
- Write second moment of the Jeans equation in cylindrical coordinates to find:

$$2v_c v_a \approx \langle v_R^2 \rangle \left[\frac{\sigma^2}{\langle v_R^2 \rangle} - \frac{3}{2} - 2\frac{\partial \ln \nu}{\partial \ln R} + \frac{1}{2}\frac{\langle v_z^2 \rangle}{\langle v_R^2 \rangle} \pm \left(\frac{\langle v_z^2 \rangle}{\langle v_R^2 \rangle} - 1 \right) \right]$$

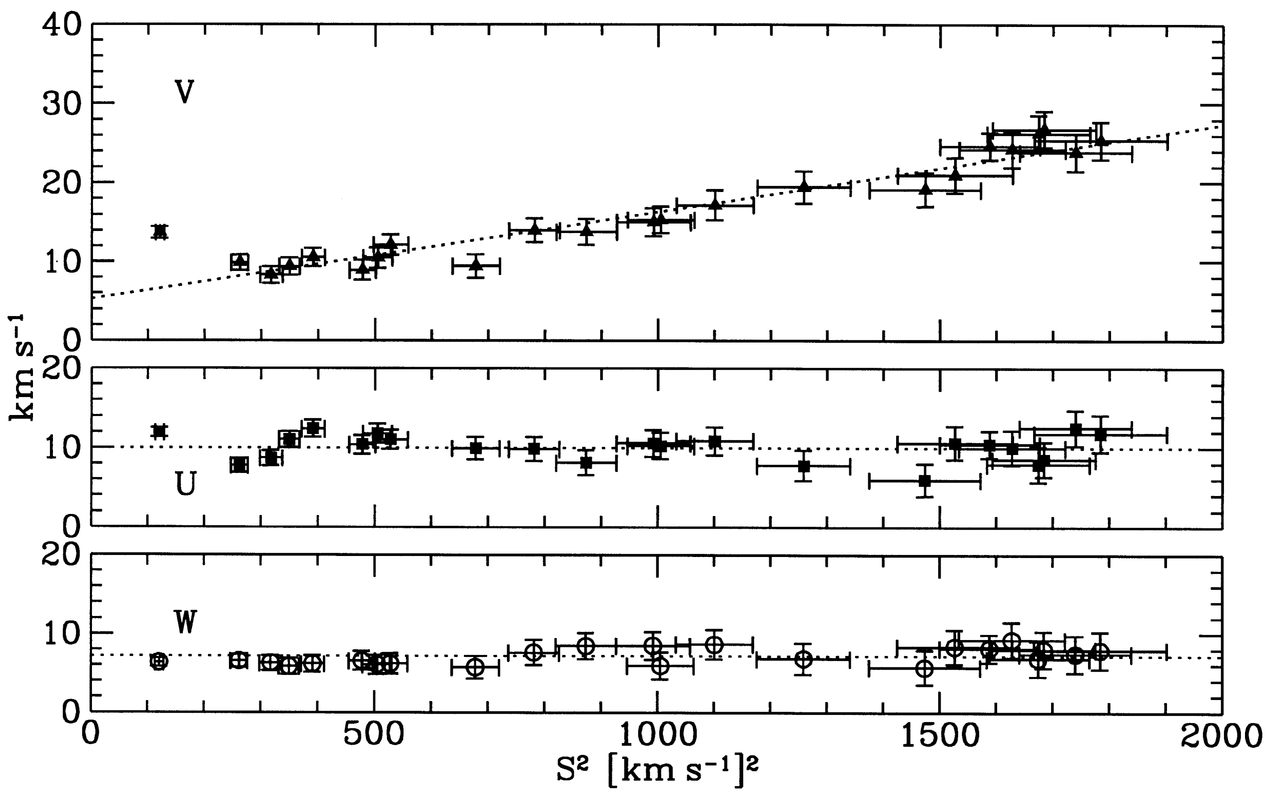
- In the Solar Neighborhood, the term in brackets is ~4

- So the lag is

$$v_a \sim 4\langle v_R^2\rangle/440\,\mathrm{km\,s^{-1}} \sim \langle v_R^2\rangle/110\,\mathrm{km\,s^{-1}}$$

  - In the absence of radial streaming, $\langle v_R^2\rangle = \sigma_R^2$
  - As σ_R^2 increases, the population lags *more* behind the circular motion — **hot populations rotate slower**
    - Energy comes out of *ordered motion* and is put into *disordered motion*

- This is why $V_\odot$ depends on the velocity dispersion of the population used to measure it — older populations have higher velocity dispersions and rotate slower



Weighing the MW disk

- Let's use the Jeans eqns to see if we can measure the mass of the MW disk in the Solar Neighborhood
  - First, select a tracer population like K dwarfs
  - Now assume that the potential is constant with time
    - Then **ν** and *f* are also constant
  - High above the plane, $\langle v_z \rangle \nu(z) \rightarrow 0$
    - Then from the first Jeans eqn, Eq. (7), $\langle v_z \rangle = 0$ everywhere

- Now we can use the second Jeans eqn, Eq. (8):

$$\frac{d}{dz}[\nu(z)\sigma_z^2] = -\frac{\partial\Phi}{\partial z}\nu(z)$$

- Next we need Poisson's equation in cylindrical coordinates:

$$4\pi G\rho(R,z) = \frac{\partial^2\Phi}{\partial z^2} + \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right)$$

- Then we can write

$$\frac{\partial\Phi}{\partial R} = \frac{v_c^2(R)}{R}$$

- and so we have

$$4\pi G\rho(R,z) = \frac{d}{dz}\left\{-\frac{1}{\nu(z)}\frac{d}{dz}[\nu(z)\sigma_z^2]\right\} + \frac{1}{R}\frac{\partial v_c^2(R)}{\partial R}$$

  - Near the Sun, the rotation curve is nearly flat, so the last term can be ignored

- To find **ρ** we need to differentiate **ν** twice, which is **very** sensitive to small errors! Let's concentrate on the *surface density* instead:

$$2\pi G\Sigma(<z) = 2\pi G\int_{-z}^{+z}\rho(z')dz' \approx -\frac{1}{\nu(z)}\frac{d}{dz}[\nu(z)\sigma_z^2]$$

- Oort was the first to try this (and derive the equations), using K giants and F dwarfs
    - He **assumed σ_z** did not vary with distance from the plane and found $\Sigma(< 700\,\mathrm{pc}) \approx 90\,M_\odot\,\mathrm{pc}^{-2}$
    - But at z>1 kpc, $\Sigma(z)$ began to **decrease**!
        - σ_z is **not** constant with distance from the plane...
- Using K *dwarfs* and allowing σ_z to vary, Kuijken & Gilmore (1991) found $\Sigma(< 1100\,\mathrm{pc}) = 71 \pm 6\,M_\odot\,\mathrm{pc}^{-2}$

- The total surface density in gas and stars in this range is ~40–55 $M_\odot$ pc^{-2}
    - But not all of the surface density determined by K&G is actually in the disk (some must be in the halo), so the *amount of dark matter in the disk is probably negligible*