Galaxy Dynamics part 2
Physics of Galaxies 2013
part 10
Orbits of stars in spherical systems

- In a *time-independent* gravitational potential, **energy is conserved**:
  
  \[ E \equiv \frac{1}{2}mv^2 + m\Phi(x) = \text{constant} \]

- because \( \partial\Phi/\partial t = 0 \)

- In a *spherical* potential, **the angular momentum is also conserved**:
  \[ \mathcal{L} \equiv x \times mv \]

- so
  \[ \frac{d\mathcal{L}}{dt} = x \times \frac{mdv}{dt} = -mx \times \nabla \Phi \]

- but the force \( \nabla \Phi \) always points towards the center, so \( \mathcal{L} \) does not change
Therefore the motion of a star is restricted to an orbital plane

and only two coordinates are needed to describe the location of the star

typically, polar coordinates in the plane \((r, \phi)\) are used to describe the motion
Orbits in an axisymmetric galaxy

- In an axisymmetric system, like a disk, we use (of course) a cylindrical coordinate system \((R, \phi, z)\) where \(z=0\) corresponds to the symmetry plane (e.g., the mid-plane of the disk).

- In an axisymmetric system, the mass distribution and therefore the potential is independent of the angular coordinate: \(\partial \Phi / \partial \phi = 0\).

- no force in the \(\phi\)-direction: **stars conserve angular momentum about the \(z\)-axis**

- neglect non-axisymmetric features like the bar and the spiral arms!
The equations of motion for a star in the disk are

\[ \frac{d^2 \mathbf{r}}{dt^2} = -\nabla \Phi \]

In each direction, using \( \mathbf{r} = R\hat{R} + z\hat{z} \), we have

\[ \frac{d^2 R}{dt^2} - R \left( \frac{d\phi}{dt} \right)^2 = -\frac{\partial \Phi}{\partial R} \]  \hspace{1cm} (1)

\[ \frac{d^2 z}{dt^2} = -\frac{\partial \Phi}{\partial z} \]  \hspace{1cm} (2)

\[ \frac{d}{dt} \left( R^2 \frac{d\phi}{dt} \right) = -\frac{\partial \Phi}{\partial \phi} = 0 \]  \hspace{1cm} (3)

Equation (3) implies conservation of angular momentum around the \( z \)-axis:

\[ L_z \equiv R^2 \frac{d\phi}{dt} = \text{constant} \]
In the $R$-direction, we can rewrite Equation (1) as

$$\frac{d^2 R}{dt^2} = R \left( \frac{d\phi}{dt} \right)^2 - \frac{\partial \Phi}{\partial R} = - \frac{\partial \Phi_{\text{eff}}}{\partial R}$$  \hspace{1cm} (4)

where the effective potential $\Phi_{\text{eff}}(R, z; L_z)$ is

$$\Phi_{\text{eff}} \equiv \Phi(R, z) + \frac{L_z^2}{2R^2}$$

If we multiply Equation (4) by $dR/dt$ and integrate with respect to $t$, we find

$$\frac{1}{2} \left( \frac{dR}{dt} \right)^2 + \Phi_{\text{eff}}(R, z; L_z) = \text{constant}$$

a sort of energy conservation law!
Therefore the effective potential $\Phi_{\text{eff}}(R, z; L_z)$ acts as a potential energy for the star’s motion in $R$ and $z$.

The effective potential is constant if

- $\frac{\partial \Phi_{\text{eff}}}{\partial R} = 0$ and thus $\frac{\partial \Phi}{\partial R} - \frac{L_z^2}{R^3} = 0$
- and $\frac{\partial \Phi_{\text{eff}}}{\partial z} = \frac{\partial \Phi}{\partial z} = 0$

The second equation is satisfied for motion in the mid-plane, so that $z=0$; combined with $dR/dt = 0$, this implies a circular orbit in the disk mid-plane.

The radius of this circular orbit is $R_g$, where

$$\left.\frac{\partial \Phi}{dR}\right|_{R_g} = \frac{L_z^2}{R^3} = R_g \left(\frac{d\phi}{dt}\right)^2$$
Since $\dot{R}^2 \geq 0$, the $L_z^2$ term in the effective potential acts as an angular momentum barrier preventing a star coming closer to $R=0$ than some perigalactic radius where $\dot{R} = 0$.

The circular orbit is that with the least energy for a given angular momentum $L_z$.

Effective potential for a star with $L_z=0.595$ in a Plummer potential.
Epicycles

- Let’s now derive approximate solutions for the equations of motion for stars on (nearly) circular orbits in the disk (symmetry) plane.

- Define a distance around \( R_g \), \( x = R - R_g \), and expand the effective potential around \((R_g,0)\):

\[
\Phi_{\text{eff}}(R,z) \sim \Phi_{\text{eff}}(R_g,0) + \frac{\partial \Phi_{\text{eff}}}{\partial R} \bigg|_{R_g,0} x + \frac{\partial \Phi_{\text{eff}}}{\partial z} \bigg|_{R_g,0} z + \cdots
\]
Let’s define two quantities:

\[ \kappa^2 = \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \bigg|_{R_g,0} \quad \text{and} \quad \nu^2 = \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \bigg|_{R_g,0} \]

Then the equations of motion become

\[
\begin{align*}
\frac{d^2 R}{dt^2} &= -\frac{\partial \Phi_{\text{eff}}}{\partial R} \quad \text{so} \quad \frac{d^2 x}{dt^2} = -x \frac{\partial^2 \Phi_{\text{eff}}}{\partial R^2} \bigg|_{R_g,0} \\
\frac{d^2 z}{dt^2} &= -\frac{\partial \Phi_{\text{eff}}}{\partial z} \quad \text{so} \quad \frac{d^2 z}{dt^2} = -z \frac{\partial^2 \Phi_{\text{eff}}}{\partial z^2} \bigg|_{R_g,0}
\end{align*}
\]

These are the equations of motion of two decoupled harmonic oscillators with frequencies \( \kappa \) and \( \nu \)

- \( \kappa \) is the **epicyclic frequency**: \( \kappa^2(R_g) = \frac{\partial^2 \Phi}{\partial R^2} \bigg|_{R_g,0} + 3 \frac{L_z^2}{R_g^4} \)
- \( \nu \) is the **vertical frequency**: \( \nu^2 = \frac{\partial^2 \Phi}{\partial z^2} \bigg|_{R_g,0} \)
The solution to the equations of motion is then

\[ x = X_0 \cos(\kappa t + \Psi) \text{ for } \kappa^2 > 0 \]
\[ z = Z_0 \cos(\nu t + \theta) \text{ for } \nu^2 > 0 \]

The motion of a star in the disk can be described as the oscillation about a guiding center moving on a circular orbit at \( R_g \)
- In order to conserve angular momentum $L_z$, the azimuthal speed must also vary:

$$\frac{d\phi}{dt} = \frac{L_z}{R^2} = \frac{\omega(R_g)R^2}{(R_g + x)^2} \approx \omega(R_g) \left(1 - \frac{2x}{R_g} + \cdots\right)$$

- Integrating, we find

$$\phi(t) = \phi_0(t) + \omega_g t - \frac{1}{R_g} \frac{2\omega}{\kappa} X_0 \sin(\kappa t + \Psi)$$

- The first two terms give the guiding center motion ($\phi_0$ is an arbitrary constant); the third term is harmonic motion with the same frequency as the $x$ oscillation but $90^\circ$ out of phase and with a different amplitude.
This motion is known as the **epicyclic motion**.

- It is retrograde because it is in the opposite sense to the guiding center’s motion: speeds the star up closer to the center and slows it down farther out.
- Note that the approximation to second order in \( z \) in the effective potential \( (\Phi_{\text{eff}} \propto z^2) \) is only valid if the density is constant in the \( z \)-direction (since \( \nabla^2 \Phi \sim \rho \)). But the disk density decreases exponentially away from the mid-plane — so the approximation is valid at most one scale height away from the plane \((z<300 \text{ pc})\). Since a good fraction of the disk stars move to greater heights, the motion in the \( z \)-direction is not well-described as a harmonic oscillation...
There is a relation between the epicyclic frequency $\kappa$ and the angular frequency $\omega$:

- First, the centrifugal force = gravitation pull, so
  \[ R\omega^2 = \frac{\partial \Phi}{\partial R} \]

- and $\omega^2 = \frac{L_z^2}{R^4}$

- so $\kappa^2 = \left( R \frac{d\omega^2}{dR} + 4\omega^2 \right)$

- In general, $\omega \leq \kappa \leq 2\omega$

- for a sphere of uniform density, $\omega(R) = \text{cnst}$ and $\kappa = 2\omega$

- for the Kepler problem (a point mass), $\omega \propto r^{-3/2}$ and $\kappa = \omega$
So the epicyclic frequency is also related to the Oort constants...

- Recall that \( A = -\frac{1}{2} R \left. \frac{d\omega}{dR} \right|_{R_0} \) and \( B = - \left( \omega + \frac{1}{2} R \frac{d\omega}{dR} \right) \bigg|_{R_0} \)

- So at the Sun, \( \kappa_0^2 = -4B(A - B) = -4B\omega_0 \)

- Using the measured value of \( B \), \( \kappa_0/\omega_0 \sim 1.3 \pm 0.2 \)

- So the Sun makes 1.3 radial oscillations in the time it takes to make one complete revolution around the Galactic Center...
The Collisionless Boltzmann Equation (CBE)

- All of these equations can actually be derived from the “collisionless Boltzmann equation” (or the Liouville equation), which describes the paths of stars in phase space.

- Let’s assume we have a large number of stars moving in a smooth potential.

- Then phase space is well-populated and a distribution function applies: $f(x, v, t) d^3 x d^3 v$ is the number of particles per unit volume in phase space $(x, v)$. 
Since the potential is smooth, neighboring particles in phase space (at the same $\mathbf{x}, \mathbf{v}$) move together, and we can use the *fluid approximation*:

$$w = (\mathbf{x}, \mathbf{v})$$

$$\dot{w} = (\mathbf{v}, -\nabla \Phi)$$

Finally, the flow is **smooth**: stars do *not* jump discontinuously from one region of phase space to another — *no collisions*

Then $f$ has to satisfy a *continuity equation*:

$$\frac{\partial f}{\partial t} + \nabla_6 \cdot (f \dot{w}) = 0$$

*phase space divergence*  *phase space “current”*
Expand equation (5):

$$\frac{\partial f}{\partial t} + f \nabla_6 \cdot (\dot{\mathbf{w}}) + \dot{\mathbf{w}} \cdot \nabla_6 f = 0$$

Consider the term $\nabla_6 \cdot (\dot{\mathbf{w}})$. This is the divergence of the flow in phase space. Let’s now show that this is incompressible:

$$\nabla_6 \cdot (\dot{\mathbf{w}}) = \sum_{i=1}^{6} \frac{\partial \dot{w}_i}{\partial w_i} = \sum_{i=1}^{3} \left( \frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right) = 0$$

$v_i$ and $x_i$ are independent

Assumed a potential, so $\partial \dot{v}_i / \partial v_i = 0$

Flow in phase space is incompressible as long as the forces do not depend on particle velocity
The result is the CBE for incompressible flow in phase space:

\[
\frac{\partial f}{\partial t} + \mathbf{w} \cdot \nabla_6 f = 0
\]

or

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \mathbf{v}} = 0
\]

This means that the **convective derivative**, the change in the local density of phase-space particles seen by an observer moving with the phase-space fluid at a velocity \( \mathbf{w} \), is zero:

\[
\frac{D f}{D t} = \frac{\partial f}{\partial t} + \sum_{i=1}^{6} \mathbf{w}_i \frac{\partial f}{\partial \mathbf{w}_i} = 0
\]

Note that this is also called the "Lagrangian derivative"
This **Liouville’s theorem**: *phase-space densities along particle trajectories are constant*

- Two key assumptions:
  - No small scale structure in the potential inside a volume element and no discontinuous jumps out of the volume element, so stars in the element are conserved: **collisionless system**
  - No frictional drag or encounters where energy and/or momentum can be exchanged with other stars: **no entropy increase**
Moments of the CBE: the Jeans Equations

- We are going to “take moments” of the CBE by multiplying $f$ by powers of $v$

- First, let’s define the space density (of the stellar population component we’re interested in) as

$$\nu \equiv \int f\,d^3v$$

- This is the zeroth moment in the velocity of the distribution function
- The **first** and **second** moments in velocity of \( f \) are

\[
\langle v_i \rangle = \frac{1}{\nu} \int v_i f \, d^3v
\]

\[
\langle v_i v_j \rangle = \frac{1}{\nu} \int v_i v_j f \, d^3v
\]

- Now, the **zeroth moment** (in velocity) of the CBE is

\[
\int \frac{\partial f}{\partial t} d^3v + \int v_i \frac{\partial f}{\partial x_i} d^3v - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3v = 0
\]

- note that here I’ve used the *summation convention*, where repeated indices are summed over
This can be rewritten as

\[ \frac{\partial}{\partial t} \int f \, d^3v + \frac{\partial}{\partial x_i} \int v_i f \, d^3v - \frac{\partial \Phi}{\partial x_i} \int \int d\nu_1 d\nu_2 \int_{\nu_3=-\infty}^{\nu_3=+\infty} \partial f = 0 \]

But the last term is 0, because

\[ f(v_i)|_{-\infty}^{+\infty} = 0 \text{ if } \lim_{v_i \to \infty} f = 0 \]

And so

\[ \frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i} (\nu v_i) = 0 \]

or

\[ \frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \nu) = 0 \]

which is the \textit{continuity equation} in \( \mathbf{x} \) space

(conservation of mass)
This can be rewritten as

\[ \frac{\partial}{\partial t} \int f \, d^3v + \frac{\partial}{\partial x_i} \int v_i f \, d^3v - \frac{\partial \Phi}{\partial x_i} \int \int dv_1dv_2 \int_{v_3=+\infty}^{v_3=-\infty} \partial f = 0 \]

But the last term is 0, because

\[ f(v_i)|_{-\infty}^{+\infty} = 0 \text{ if } \lim_{v_i \to \infty} f = 0 \]

And so

\[ \frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i} (\nu v_i) = 0 \]

or

\[ \frac{\partial \nu}{\partial t} + \nabla \cdot (\nu v) = 0 \] (7)

which is the continuity equation in \( \mathbf{x} \) space (conservation of mass)
The first moment in velocity of the CBE is

\[ \int v_j \frac{\partial f}{\partial t} d^3v + \int v_i v_j \frac{\partial f}{\partial x_i} d^3v - \frac{\partial \Phi}{\partial x_i} \int v_i \frac{\partial f}{\partial v_i} d^3v = 0 \]

Now,

\[ \int v_j \frac{\partial f}{\partial v_i} d^3v = \int \int dv_1 dv_2 \int_{v_3=+\infty} \int_{v_3=-\infty} \partial f - \int \left( \frac{\partial v_j}{\partial v_i} \right) f d^3v \]

\[ = 0 - \delta_{ij} \nu \]

So

\[ \frac{\partial}{\partial t} (\nu \langle v_j \rangle) + \frac{\partial}{\partial x_i} (\nu \langle v_i v_j \rangle) + \nu \frac{\partial \Phi}{\partial x_j} = 0 \]
Subtracting off the zeroth moment, we have

\[
\nu \frac{\partial \langle v_j \rangle}{\partial t} - \langle v_i \rangle \frac{\partial}{\partial x_i} \left( \nu \langle v_j \rangle \right) + \frac{\partial}{\partial x_i} \left( \nu \langle v_i v_j \rangle \right) + \nu \frac{\partial \Phi}{\partial x_j} = 0
\]

We usually rewrite this in terms of the velocity dispersion tensor, the velocity dispersion about the mean (streaming) motion:

\[
\sigma^2_{ij} = \langle (v_i - \langle v_i \rangle)(v_j - \langle v_j \rangle) \rangle = \langle v_i v_j \rangle - \langle v_i \rangle \langle v_j \rangle
\]
Substituting and rearranging, we finally have

\[
\nu \frac{\partial \langle v_j \rangle}{\partial t} + \nu \langle v_i \rangle \frac{\partial \langle v_j \rangle}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i}
\]

(8)

- The left-hand side of this equation is the Lagrangian derivative of the momentum: \( \nu Dv/Dt \)
- The right-hand side is the sum of the gravity and a stress term, which is the anisotropic pressure
- This equation is a force equation: the conservation of momentum along particle trajectories
Since the velocity dispersion tensor $\sigma_{ij}$ is symmetric, it can be diagonalized. This diagonalized tensor is the **velocity ellipsoid**, which we met earlier...

So what use are the Jeans equations (moments of the CBE)?

- Can relate observables like $\nu$, $\mathbf{v}$, and $\sigma_{ij}^2$ to the gravitational potential $\partial \Phi / \partial x_i$
- they give us a way to “weigh” galaxies
But!

- Jeans equations describe a massless tracer population in an external potential — need to add Poisson’s equation to get the potential from the (total) mass density $\rho$: Jeans equations are incomplete.

- There is no “equation of state” relating $v$ to $\sigma$ (like the ideal gas law relates $\rho$ and $T$ in a gas). You have to assume $\sigma_{ij}$, and thus $f(v)$ — every different assumption leads to a different solution: Jeans equations depend on $f(v)$ and are non-unique.

- The equations never close; have always to assume a higher-order tensor, i.e., some form for $f(v)$. 
Asymmetric drift: an application of the Jeans eqn

- Let’s evaluate the quantity \( v_a \equiv v_c - \langle v_\phi \rangle \)
- the amount a population lags behind the circular velocity

- Write second moment of the Jeans equation in cylindrical coordinates to find:

\[
2v_cv_a \approx \langle v_z^2 \rangle \left[ \frac{\sigma^2}{\langle v_z^2 \rangle} - \frac{3}{2} - 2 \frac{\partial \ln \nu}{\partial \ln R} + \frac{1}{2} \frac{\langle v_z^2 \rangle}{\langle v_z^2 \rangle} \pm \left( \frac{\langle v_z^2 \rangle}{\langle v_z^2 \rangle} - 1 \right) \right]
\]

- In the Solar Neighborhood, the term in brackets is \( \sim 4 \)
• So the lag is

\[ \nu_a \sim 4\langle v^2_R \rangle / 440 \text{ km s}^{-1} \sim \langle v^2_R \rangle / 110 \text{ km s}^{-1} \]

• In the absence of radial streaming, \[ \langle v^2_R \rangle = \sigma^2_R \]

• As \( \sigma^2_R \) increases, the population lags more behind the circular motion — **hot populations rotate slower**

• Energy comes out of *ordered motion* and is put into *disordered motion*
This is why \( V_\odot \) depends on the velocity dispersion of the population used to measure it — older populations have higher velocity dispersions and rotate slower.

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**Figure 3.** The components \( U, V \) and \( W \) of the solar motion with respect to stars with different colour \( B^\text{1} V \). Also shown is the variation of the dispersion \( S^2 \) with colour.

**Figure 4.** The dependence of \( U, V \) and \( W \) on \( S^2 \). The dotted lines correspond to the linear relation fitted (\( V \)) or the mean values (\( U \) and \( W \)) for stars bluer than \( B^\text{1} V = 0 \).
Weighing the MW disk

- Let’s use the Jeans eqns to see if we can measure the mass of the MW disk in the Solar Neighborhood
  - First, select a tracer population like K dwarfs
  - Now assume that the potential is constant with time
    - Then $v$ and $f$ are also constant
  - High above the plane, $\langle v_z \rangle \nu(z) \rightarrow 0$
    - Then from the first Jeans eqn, Eq. (7), $\langle v_z \rangle = 0$ everywhere
Now we can use the second Jeans eqn, Eq. (8):
\[
\frac{d}{dz} [\nu(z) \sigma_z^2] = - \frac{\partial \Phi}{\partial z} \nu(z)
\]

Next we need Poisson’s equation in cylindrical coordinates:
\[
4\pi G \rho(R, z) = \frac{\partial^2 \Phi}{\partial z^2} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \frac{\partial \Phi}{\partial R} \right)
\]

Then we can write
\[
\frac{\partial \Phi}{\partial R} = \frac{v_c^2(R)}{R}
\]
and so we have

$$4\pi G \rho(R, z) = \frac{d}{dz} \left\{ -\frac{1}{\nu(z)} \frac{d}{dz} \left[ \nu(z) \sigma_z^2 \right] \right\} + \frac{1}{R} \frac{\partial v_c^2(R)}{\partial R}$$

Near the Sun, the rotation curve is nearly flat, so the last term can be ignored.

To find $\rho$ we need to differentiate $\nu$ twice, which is very sensitive to small errors! Let’s concentrate on the surface density instead:

$$2\pi G \Sigma(< z) = 2\pi G \int_{-z}^{+z} \rho(z') dz' \approx -\frac{1}{\nu(z)} \frac{d}{dz} \left[ \nu(z) \sigma_z^2 \right]$$

• Oort was the first to try this (and derive the equations), using K giants and F dwarfs

• He **assumed** $\sigma_z$ did not vary with distance from the plane and found $\Sigma(< 700 \text{ pc}) \approx 90 \, M_\odot \text{ pc}^{-2}$

• But at $z > 1 \text{ kpc}$, $\Sigma(z)$ began to **decrease**!

  • $\sigma_z$ is **not** constant with distance from the plane...

• Using K **dwarfs** and allowing $\sigma_z$ to vary, Kuijken & Gilmore (1991) found $\Sigma(< 1100 \text{ pc}) = 71 \pm 6 \, M_\odot \text{ pc}^{-2}$
The total surface density in gas and stars in this range is \( \sim 40\text{--}55 \, M_\odot \, \text{pc}^{-2} \).

But not all of the surface density determined by K&G is actually in the disk (some must be in the halo), so the amount of dark matter in the disk is probably negligible.