

1.5 Density-potential pairs & orbits

Recall that the first step in constructing a galaxy is to pick a gravitational potential. Since differentiation is generally much more manageable than integration, it should be no surprise that we rely on Poisson's Equation to get the density distribution from the potential ($\Phi \rightarrow \rho$ via differentiation rather than $\rho \rightarrow \Phi$ via integration). We shall loosely speak of a density-potential pair being a “model” for a galaxy, but such models are incomplete. As we saw in Figure 1.3 there are many ways to populate a given density with different orbits. A complete description requires not just a physical density $\rho(\vec{x})$, but a phase space density $f(\vec{x}, \vec{p})$ (# stars/unit volume/unit momentum).

Different density-potential pairs are good for different purposes. Some have the virtue of simplicity or some special symmetry. Others permit one to write the phase space density in closed form, often a function of only a limited number of variables: $f = f(E)$, or $f = f(E, L^2)$ or $f = f(E, L_z)$, where E , L and L_z are respectively the total energy of the orbit, its angular momentum and its angular momentum around the z-axis. Yet others have the property that orbits are self-similar, meaning that an orbit at any radius can be scaled by an arbitrary factor producing another legitimate orbit. This is especially useful when generating orbit libraries.

Before proceeding we will want to tighten up our language a bit. We shall use *ellipsoid* to mean something whose equi-something contours are elliptical in cross-section but which generically have three unequal axes. A *spheroid* has also elliptical cross-sections but has two equal axes. *Oblate spheroids* have their long axes equal while *prolate spheroids* have their short axes equal.

We present the following list of density-potential pairs so that our reader is not taken by surprise when he encounters one in a talk or in the literature. He might even ask “Please remind me what special property of the something-something potential makes it appropriate for the problem at hand” and then say “ah yes” upon hearing the answer.

1.5.1 spherical potentials:

- a) **point mass:** The Kepler potential with

$$\Phi(r) = -\frac{GM}{r}. \quad (1.41)$$

The energy of an object with mass μ is a function only of the elliptical orbit’s semi-major axis

a:

$$E(a) = -\frac{GM\mu}{2a}. \quad (1.42)$$

- b) **homogeneous sphere:** A constant density sphere of mass $M = \frac{4}{3}\pi\rho R^3$ gives orbits within the sphere with constant periods

$$T = \sqrt{\frac{3\pi}{G\rho}}. \quad (1.43)$$

Formulae like this are common throughout astrophysics. The typical dynamical time scale T_{dyn} must vary as $(G\rho)^{-1/2}$. The potential outside a homogeneous sphere is the same as that of the point mass $\Phi(r) = -GM/r$ while for $r < R$,

$$\Phi(r) = -2\pi G\rho \left(R^2 - \frac{r^2}{3} \right). \quad (1.44)$$

- c) **isochrone:** This potential has the interesting property that the orbital period is a function only of the energy of the orbit. Recall that the Kepler potential has a similar degeneracy – the energy is a function only of the semi-major axis.

$$\Phi(r) = -\frac{GM}{b + \sqrt{b^2 + r^2}}, \quad (1.45)$$

giving a central density of

$$\rho_o = \frac{3M}{16\pi b^3} \quad (1.46)$$

- d) **modified Hubble:** The modified Hubble law starts with a surface brightness profile that at one time was thought to be a close fit to those observed in for elliptical galaxies,

$$I(r) = \frac{2j_0a}{1 + (R/a)^2}, \quad (1.47)$$

where j_0 is a central surface brightness and a is a scale length. It is a variant of a similar one called the *Hubble-Reynolds Law*. To first order it matches the profile for a non-singular isothermal sphere. The scale length plays the role of the core radius of the non-singular isothermal sphere. The associated analytic potential is somewhat cumbersome, but it is still of some use (as we shall see below) for clusters of galaxies.

- e) **power-law density:** These models have simple power-law expressions for the density profile with

$$\rho(r) = \rho_o \left(\frac{r_o}{r} \right)^\alpha. \quad (1.48)$$

A special case of the power-law density potential is the *singular isothermal sphere*. It has a constant velocity dispersion and a density profile $\rho \propto r^{-2}$. This gives our old friend, the logarithmic potential described in Sec. (1.2). It has a flat rotation curve and infinite mass and radius. The logarithmic potentials produce self-similar orbits. There are therefore theoretical as well as practical reasons for choosing them.

- f) **Dehnen (with special cases Jaffe and Hernquist):** These have “cuspy” density profiles for small r and a power law of the form $\rho \propto r^{-4}$ as $r \rightarrow \infty$. The phase-space distribution function can be written in closed form as a function of energy and angular momentum, $f(E, L^2)$.
- g) **King models:** Starting with the isothermal sphere, King introduced an energy cutoff to the distribution function that limits the size of the system. The associated *truncation radius* r_T is often suggestively called a tidal radius.
- h) **Navarro-Frenk-White:** This is an empirical fit to the observed density profiles in gravitational N-body experiments,

$$\rho(r) = \frac{4\rho_s}{(r/r_s)(1+r/r_s)^2} , \quad (1.49)$$

where r_s is a scale length. It is a member of the larger family

$$\rho(r) = \frac{2^{3-\gamma}\rho_s}{(r/r_s)^\gamma(1+r/r_s)^{(3-\gamma)}} , \quad (1.50)$$

of which the $\gamma = 1.5$ case is called the *Moore profile*. Astronomy is not immune to fashion and these days the NFW and Moore profiles are quite fashionable, both as models for galaxy halos and for clusters of galaxies.

1.5.2 axisymmetric potentials

- a) **Plummer-Kuzmin:** also known as “Toomre’s model 1.” In cylindrical coordinates (R, z, ϕ) it has

$$\Phi(R, z) = \frac{-GM}{\sqrt{R^2 + (a + |z|)^2}} , \quad (1.51)$$

giving a infinitesimally thin sheet of mass density at $z = 0$ with mass surface density

$$\Sigma(R) = \frac{aM}{2\pi(R^2 + a^2)^{3/2}} . \quad (1.52)$$

- b) **Miyamoto:** A combination of the Plummer-Kuzmin model and the spherical Plummer model that is often used for globular clusters,

$$\Phi(R, z) = \frac{-GM}{\sqrt{R^2 + (a + \sqrt{b^2 + z^2})^2}} . \quad (1.53)$$

- c) **“cored” logarithmic:** A modified logarithmic potential that avoids a singularity at $R = 0$ and scales the z axis by the factor q so as to give a spheroidal potential, with $q < 1$ in the oblate case and $q > 1$ in the prolate case:

$$\Phi(R, z) = \frac{1}{2}v_o^2 \ln \left(R_c^2 + R^2 + \frac{z^2}{q^2} \right) . \quad (1.54)$$

In the limit of vanishing core radius R_c , the equipotentials all have the same spheroidal shape and the equidensity contours likewise all have the same shape (though not spheroidal). The orbits are therefore self-similar.

d) **Mestel disk:** As in Sec. (1.2), we write this in terms of spherical coordinates,

$$\Phi(r, \theta) = v_c^2 \left[\ln \frac{r}{R_c} + \ln \frac{1 + |\cos \theta|}{2} \right]. \quad (1.55)$$

e) **self-similar logarithmic:** As its name implies, the both the equipotential surfaces and the equidensity contours all have the same shape. It is separable in r and θ , with

$$\Phi(r, \theta) = v_o^2 \ln[(r \cdot Q(\theta))] \quad \text{and} \quad (1.56)$$

$$\rho(r, \theta) = \rho_o \left(\frac{r_o}{r} \right)^2 S(\theta), \quad (1.57)$$

where $Q(\theta)$ and $S(\theta)$ are arbitrary functions for the potential and density (but of course related to each other via Poisson's equation). The spheroidal logarithmic potentials describe above are special cases of this.

f) **homoeoids:** A homoeoid is a uniform density shell of finite thickness whose inner and outer surfaces are similar spheroids. A thin homoeoid is the limiting case where the inner and outer shells approach each other. Thin homoeoids have the remarkable property their external equipotentials are spheroids that are confocal with the thin homoeoid, while their internal potentials are constant. This can be used to generate the potentials of arbitrary spheroidal mass distributions. Many of the results for spheroidal systems are most simply expressed in *spheroidal coordinates*. If one draws nested confocal spheroids, there is a corresponding family of confocal hyperboloids of revolution. One coordinate is constant on the ellipses (playing the role of the radius in spherical coordinates) and the other is constant on the hyperboloids of revolution, playing the role of the polar angle.

1.5.3 non-axisymmetric potentials and the multipole expansion

The logarithmic potential and homoeoid above are straightforwardly generalized to non-axisymmetric cases. In the logarithmic case of the potential, the density and the functions Q and S are then functions of r, θ and ϕ . The homoeoids are by extension ellipsoidal rather than spheroidal.

For more general three-dimensional systems, last resort is to expand the potential in spherical harmonics:

$$\Phi(r, \theta, \phi) = -4\pi G \sum \frac{Y_{lm}(\theta, \phi)}{2l+1} \left[\frac{1}{r^{l+1}} \int_0^r \rho_{lm}(a) a^{l+2} da + r^l \int_r^\infty \frac{\rho_{lm}(a)}{a^{l-1}} da \right], \quad (1.58)$$

where Y_{lm} are the spherical Legendre functions and

$$\rho_{lm}(a) = \int Y_{lm}^*(\theta, \phi) \rho(a, \theta, \phi) d\Omega. \quad (1.59)$$

The multipole expansion method is frequently used in N-body simulations because of the computationally efficient methods for solving the Laplacian with spherical harmonics.