25 Lecture 25: Galaxies: Analytic Models

"Science is simply common sense at its best that is, rigidly accurate in observation, and merciless to fallacy in logic."

Thomas Henry Huxley

The Big Picture: Last time we showed that individual stellar encounters are unimportant in the dynamics of the galaxy, which justifies the mean-field approximation and the use of the collisionless Boltzmann equation. Today we derive the collisionless Boltzmann equation in the context of galaxies, formulate the self-consistent problem and outline a few analytic approaches to solving it.

The study of galactic systems — the dynamics, kinematics, morphology — is a major tool in comprehending some of the key issues in astrophysics relating to the origin, evolution and structure of the Universe.

In modeling of galactic systems, we move from the simplest approximations to galaxy shapes (spherical — 1 dof) to more general (axisymmetric — 2 dof; and triaxial — 3 dof). However, we first must establish which equations govern the dynamics of galactic systems.

The Collisionless Boltzmann Equation

Earlier, we have demonstrated that in galaxies the stellar encounters are unimportant; in other words, the mean-free path between collisions is considerably (orders of magnitude!) longer than the age of the Universe. This justifies the collisionless approximation and the use of the *collisionless Boltzmann equation* (also known as the Vlasov equation).

Imagine a large number of stars moving under the influence of a smooth potential $\Phi(\mathbf{x}, t)$. At any time t, a full description of the state of any collisionless system is given by specifying the number of stars $f(\mathbf{x}, \mathbf{v}, t)d^3\mathbf{x}d^3\mathbf{v}$ having positions in the small volume $d^3\mathbf{x}$ centered on $\mathbf{x}$ and velocities in the small range $d^3\mathbf{v}$ centered on $\mathbf{v}$. The quantity $f(\mathbf{x}, \mathbf{v}, t)$ is called the *distribution function* or *phase-space density* of the system. Clearly $f \geq 0$ everywhere.

If we know the initial coordinates and velocities of every star, Newton's laws enable us to evaluate their positions and velocities at any later time. Thus, given $f(\mathbf{x}, \mathbf{v}, t_0)$, it should be possible to calculate $f(\mathbf{x}, \mathbf{v}, t)$ for any t using only the information that is contained in $f(\mathbf{x}, \mathbf{v}, t_0)$. Now, consider the flow of points in phase space that arises as stars move along their orbits. The coordinates in phase-space are

$$(\mathbf{x}, \mathbf{v}) \equiv \mathbf{w} \equiv (w_1, ..., w_6), \tag{513}$$

so that the velocity of this flow can be written as

$$\dot{\mathbf{w}} = (\dot{\mathbf{x}}, \dot{\mathbf{v}}) = (\dot{\mathbf{x}}, -\nabla\Phi), \tag{514}$$

where we have used from the Hamiltonian formulation $\dot{\mathbf{v}} = -\nabla\Phi$.

A characteristic of the flow described by $\dot{\mathbf{w}}$ is that it conserves stars: in the absence of encounters stars do not jump from one point in phase-space to another, but rather drift smoothly through space. Therefore, the density of stars $f(\mathbf{w}, t)$ satisfies a continuity equation analogous to that satisfied by the density $\rho(\mathbf{x}, t)$ of the ordinary fluid flow:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \frac{\partial (f \dot{w}_i)}{dw_i} = 0. \tag{515}$$

The physical content of this equation can be seen by integrating it over some volume of phase space. The first term then describes the rate at which the collection of stars inside this volume is increasing, while an application of the divergence theorem shows that the second term describes the rate at which stars flow out of this volume.

The flow described by $\dot{\mathbf{w}}$ is very special, because it has the property that

$$\sum_{i=1}^{6} \frac{\partial \dot{w}_i}{dw_i} = \sum_{j=1}^{3} \frac{\partial v_j}{dx_j} + \frac{\partial \dot{v}_j}{dv_j} = \sum_{j=1}^{3} -\frac{\partial}{dv_j}\left(\frac{\partial \Phi}{dx_j}\right) = 0. \tag{516}$$

Here $(\partial v_j/\partial x_j) = 0$ because v_i and x_i are independent coordinates of phase-space, and the last step follows because $\nabla\Phi$ does not depend on velocities. If we use eq. (516) to simplify eq. (515), we obtain the *collisionless Boltzmann equation* (also known as the Vlasov equation):

$$\begin{aligned}
\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \frac{\partial (f\dot{w}_i)}{\partial w_i} &= 0 \\
\frac{\partial f}{\partial t} + \sum_{i=1}^{6} \left(f\frac{\partial \dot{w}_i}{\partial w_i} + \dot{w}_i \frac{\partial f}{\partial w_i}\right) &= 0 \\
\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\dot{x}_i\frac{\partial f}{\partial x_i} + \dot{v}_i \frac{\partial f}{\partial v_i}\right) &= 0 \\
\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(v_i\frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i}\right) &= 0 \qquad (517)
\end{aligned}$$

or, in vector notation

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} = 0. \tag{518}$$

Equation (518) is the fundamental equation of stellar dynamics.

The meaning of the collisionless Boltzmann equation can be clarified by extending to six diversions the concept of the convective derivative. We define

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \sum_{i=1}^{6} \dot{w}_i \frac{\partial f}{\partial w_i}. \tag{519}$$

df/dt represents the rate of change of density of phase points as seen by an observer who moves through phase-space with a star at velocity $\dot{\mathbf{w}}$. The collisionless Boltzmann equation is then simply

$$\frac{df}{dt} = 0. \tag{520}$$

In words, the flow of stellar phase points through phase-space is incompressible; the phase-space density f around the phase point of a given star always remains the same.

The Self-Consistent Problem

The collisionless Boltzmann equation does not provide the closed system of equation. In order to have a closed system of equation, we must have as many equations as we have quantities. Here, it means that we must relate Φ and f. The Poisson equation

$$\Delta\Phi(\mathbf{x}, t) = 4\pi G\rho(\mathbf{x}, t) \tag{521}$$

relates the mass-density $\rho(\mathbf{x},t)$ to the distribution function $f(\mathbf{x},\mathbf{v},t)$. Finally, the potential $\Phi(\mathbf{x},t)$ and density $\rho(\mathbf{x},t)$ are related as

$$\rho(\mathbf{x},t) = \int f(\mathbf{x},\mathbf{v},t) d^3\mathbf{v}, \tag{522}$$

which provides the link $\Phi \leftrightarrow \rho \leftrightarrow f$, and closes the system of equations. Solving the system of equations:

$$\begin{aligned} \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \mathbf{v}} &= 0, \\ \rho(\mathbf{x},t) &= \int f(\mathbf{x},\mathbf{v},t) d^3\mathbf{v}, \\ \Delta\Phi(\mathbf{x},t) &= 4\pi G\rho(\mathbf{x},t) \end{aligned} \tag{523}$$

simultaneously is called the *self-consistent problem.*

Integrals of Motion and Jeans Theorem

An *integral of motion* $I(\mathbf{x},\mathbf{v})$ is any function of the phase-space coordinates $(\mathbf{x},\mathbf{v})$ that is constant along any orbit:

$$I[\mathbf{x}(t_1),\mathbf{v}(t_1)] = I[\mathbf{x}(t_2),\mathbf{v}(t_2)], \tag{524}$$

or

$$\frac{d}{dt}I[\mathbf{x}(t_1),\mathbf{v}(t_1)] = 0 = \frac{\partial I}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial I}{\partial \mathbf{v}}\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v}\frac{\partial I}{\partial \mathbf{x}} - \nabla\Phi\frac{\partial I}{\partial \mathbf{v}}, \tag{525}$$

which satisfies the collisionless Boltzmann equation. This leads to the following theorems.

Jeans theorem. *Any steady-state solution of the collisionless Boltzmann equation depends on the phase-space coordinates only through integrals of motion in the galactic potential, and any function of the integrals yields a steady-state solution of the collisionless Boltzmann equation.*

Strong Jeans theorem. *The DF of a steady-state galaxy in which almost all orbits are regular with incommensurate frequencies may be presumed to be a function only of the three independent isolating integrals.*

In other words, the Jeans theorem tells us that if I_1,..., I_5 are five independent integrals of motion in a given potential, then any DFs of the forms $f(I_1)$, $f(I_1,I_2)$, ..., $f(I_1,...,I_5)$ are solutions of the collisionless Boltzmann equation. The strong Jeans theorem tells us that if the potential is regular (*integrable*), for all practical purposes any time-independent galaxy may be represented by a solution of the form $f(I_1,I_2,I_3)$, where I_1, I_2 and I_3 are any three independent integrals of motion.

For example, in a spherical system (1 dof), the DF is a function of energy: $f(E)$; in an (integrable) axisymmetric system (2 dof), the DF is a function of energy and a z-component of the angular momentum $f(E,L_z)$; and in a (integrable) triaxial systems (3 dof), the DF is a function of energy and two more integrals of motion: $f(E,I_2,I_3)$. In general, integrals of motion I_2 and I_3 are not known, except in very special cases (of limited physical importance). For equilibrium models $df/dt = 0$, so the energy is conserved, and therefore an integral of motion.

So, how does one construct DFs for galactic models?

Analytic Solutions to the Self-Consistent Problem

The DFs for galactic models can be obtained analytically only for a few special cases. These special cases are important phenomenologically and pedagogically, as they offer a “peek” into the dynamics of galaxies. However, their physical relevance is limited, because they represent either simple 1 dof models (spheres), or density distributions which give poor fits to the observed profiles.

From f to ρ.
As a simple spherical model (1 dof), one can start with the predefined DF $f(E)$ and compute the corresponding ρ. This is the most straightforward method. The drawback of this approach, however, is that the properties of the resulting density distribution are not adjustable to fit the observed profiles.

We start with an assumed form of the DF f, integrate to obtain ρ, and solve the Poisson equation to get the corresponding Φ.

Define relative potential and relative energy, respectively:

$$\begin{aligned} \Psi &\equiv -\Phi + \Phi_0, \\ \epsilon &\equiv -E + \Phi_0 = \Psi - \frac{1}{2}v^2, \end{aligned} \tag{526}$$

and assume the DF of the following form:

$$f(\epsilon) = \begin{cases} F\epsilon^{n-3/2} & \epsilon > 0, \\ 0 & \epsilon \leq 0, \end{cases} \tag{527}$$

where F is a constant. Then the mass-density is computed by integrating over velocities [see eq. (522)]:

$$\rho(\mathbf{x}) = \int_0^\infty f(\epsilon) d^3\mathbf{v} = \int_0^\infty f\left(\Psi - \frac{1}{2}4\pi v^2\right) v^2 dv = 4\pi F \int_0^{\sqrt{2\Psi}} \left(\Psi - \frac{1}{2}v^2\right)^{n-3/2} v^2 dv, \tag{528}$$

where we have used $d^3\mathbf{v} = 4\pi v^2$. After introducing the variable θ, such that $v^2 = 2\Psi\cos^2\theta$, we obtain

$$\begin{aligned} \rho(\mathbf{x}) &= 4\pi F \int_0^{\pi/2} \Psi^{n-3/2}\left(1-\cos^2\theta\right)^{n-3/2}\left(2\Psi\cos^2\theta\right)\left(\sqrt{2\Psi}\sin\theta d\theta\right) = \\ &= 8\sqrt{2}\pi F\Psi^n \int_0^{\pi/2} \sin^{2n-2}\theta\cos^2\theta d\theta \\ &= 8\sqrt{2}\pi F\Psi^n \left[\int_0^{\pi/2} \sin^{2n-2}\theta d\theta - \int_0^{\pi/2}\sin^{2n}\theta d\theta\right] \\ \Longrightarrow \qquad \rho(\mathbf{x}) &= c_n\Psi^n, \end{aligned} \tag{529}$$

where

$$c_n = \frac{(2\pi)^{3/2}\left(n-\frac{3}{2}\right)!}{n!}F. \tag{530}$$

For c_n to be finite, $n > 1/2$.

We now solve the Poisson equation by substituting the eqs. (526) and (529) into the eq. (521) expressed in spherical coordinates:

$$\begin{aligned} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) &= 4\pi G\rho \\ -\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) &= 4\pi G c_n\Psi^n. \end{aligned} \tag{531}$$

Now let

$$\begin{aligned} s &\equiv \frac{r}{b}, \\ \varphi &\equiv \frac{\Psi}{\Psi_0}, \\ b &\equiv \frac{1}{\sqrt{4\pi G\Psi_0^{n-1}c_n}}. \end{aligned} \tag{532}$$

Then we arrive at

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\varphi}{ds}\right) = \begin{cases} -\varphi^n & \varphi > 0, \\ 0 & \varphi \le 0, \end{cases} \tag{533}$$

which is the Lane-Emden equation for polytropes! Again, this second-order ODE is to be solved with the initial conditions:

1. $\varphi(0) = 1$ by definition;
2. $\left.\frac{d\varphi}{ds}\right|_{s=0} = 0$: no gravitational force at the center.

Table 8: Properties of the solutions to the Lane-Emden equation [$\gamma = (n+1)/n$].

Lane-Emden index n	**radius**	**mass**	**polytropic index γ**
$1 \le n < 5$	finite	finite	$6/5 < \gamma \le \infty$
$5 \le n < \infty$	infinite	finite	$1 < \gamma \le 6/5$
$n = \infty$	infinite	infinite	$\gamma = 1$

One of the popular early simple models for the DF in a spherical galaxy is the solution to the Lane-Emden equation with $n = 5$. It is called the *Plummer model*:

$$\begin{aligned} f(\epsilon) &= F\epsilon^{7/2}, \\ \Phi(r) &= -\frac{GM}{\sqrt{r^2+b^2}}, \\ \rho(r) &= \frac{3Mb^2}{4\pi\left(r^2+b^2\right)^{5/2}}. \end{aligned} \tag{534}$$

From ρ to f.
Another simple spherical model (1 dof) is obtained by starting with the predefined density $\rho(r)$ and compute the corresponding DF $f(E)$.

We first invert the integral for ρ in terms of f, in order to get f in terms of ρ:

$$\begin{aligned} \rho(r) &= \int_0^{\sqrt{2\Psi(r)}} f(\epsilon)4\pi v^2 dv \qquad\qquad \epsilon = \Psi(r) - \frac{1}{2}v^2, \quad d\epsilon = -vdv \\ \rho(\Psi) &= 2\pi\sqrt{2}\int_{\epsilon=0}^{\Psi} f(\epsilon)\sqrt{\Psi-\epsilon}\; d\epsilon \\ \frac{d\rho(\Psi)}{d\Psi} &= 4\pi\sqrt{2}\int_{\epsilon=0}^{\Psi}\frac{f(\epsilon)}{\sqrt{\Psi-\epsilon}}\; d\epsilon \end{aligned} \tag{535}$$

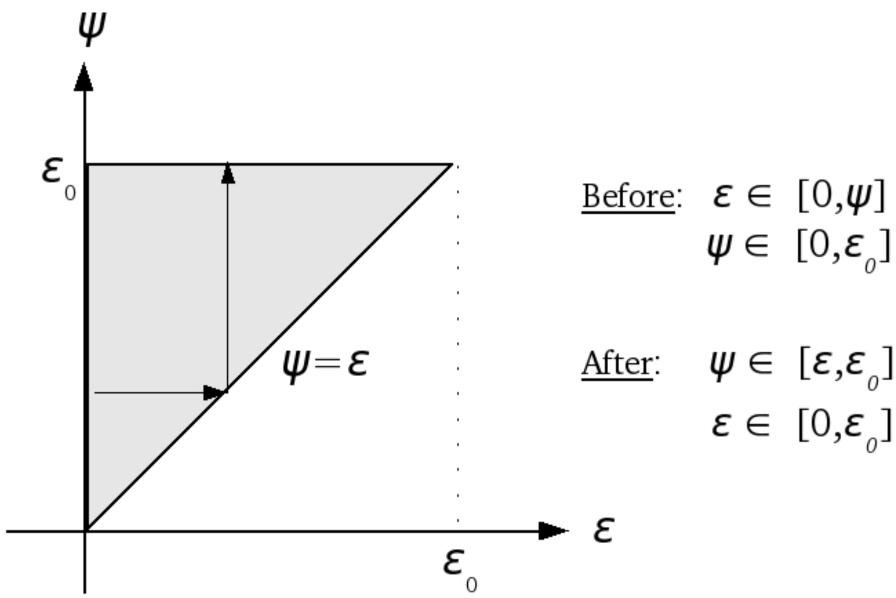

Figure 43: Region of integration for the integral in the eq. (536).

The last line represents the Abel integral equation, which can be solved explicitly. Multiply both sides by $\frac{1}{\sqrt{\epsilon_0-\Psi}}$ and integrate with respect to Ψ from 0 to ϵ_0:

$$\begin{aligned}\int_0^{\epsilon_0}\frac{\rho'(\Psi)}{\sqrt{\epsilon_0-\Psi}}d\Psi &= 2\pi\sqrt{2}\int_0^{\epsilon_0}\frac{d\Psi}{\sqrt{\epsilon_0-\Psi}}\int_0^{\Psi}\frac{f(\epsilon)}{\sqrt{\Psi-\epsilon}}\,d\epsilon \\ &= 2\pi\sqrt{2}\int_0^{\epsilon_0}f(\epsilon)d\epsilon\int_{\epsilon}^{\epsilon_0}\frac{d\Psi}{\sqrt{(\epsilon_0-\Psi)\,(\Psi-\epsilon)}}. \end{aligned} \tag{536}$$

After setting $\Psi=\epsilon+(\epsilon_0-\epsilon)\sin^2\chi$, the inner integral becomes

$$\int_0^{\pi/2}\frac{2(\epsilon_0-\epsilon)\sin\chi\cos\chi}{\sqrt{(\epsilon_0-\epsilon)\cos^2\chi(\epsilon_0-\epsilon)\sin^2\chi}}\,d\chi = 2\frac{\pi}{2}=\pi, \tag{537}$$

so the integral in eq. (536) becomes

$$\begin{aligned}\int_0^{\epsilon_0}f(\epsilon)d\epsilon &= \frac{1}{2\sqrt{2}\pi^2}\int_0^{\epsilon_0}\frac{\rho'(\Psi)}{\sqrt{\epsilon_0-\Psi}}d\Psi, \\ \Longrightarrow \qquad f(\epsilon_0) &= \frac{1}{2\sqrt{2}\pi^2}\frac{d}{d\epsilon_0}\int_0^{\epsilon_0}\frac{\rho'(\Psi)}{\sqrt{\epsilon_0-\Psi}}d\Psi. \end{aligned} \tag{538}$$

Now integrate the integral in the eq. (538) by parts:

$$\begin{aligned}\int_0^{\epsilon_0}\frac{\rho'(\Psi)}{\sqrt{\epsilon_0-\Psi}}d\Psi &= \left[\rho'(\Psi)\left(-2\sqrt{\epsilon_0-\Psi}\right)\right]_0^{\epsilon_0}-\int_0^{\epsilon_0}\rho''(\Psi)\left(-2\sqrt{\epsilon_0-\Psi}\right)d\Psi \\ &= 2\rho'(0)\sqrt{\epsilon_0}+2\int_0^{\epsilon_0}\rho''(\Psi)\sqrt{\epsilon_0-\Psi}d\Psi, \end{aligned} \tag{539}$$

so

$$f(\epsilon_0) = \frac{1}{2\sqrt{2}\pi^2}\left[\frac{\rho'(0)}{\sqrt{\epsilon_0}} + \int_0^{\epsilon_0} \frac{\rho''(\Psi)}{\sqrt{\epsilon_0 - \Psi}}\, d\Psi\right] \tag{540}$$

Equations (538) and (540) are two variants of *Eddington's formula*.

We now apply Eddington's formula [top line of eq. (538)] to the density used in the approach "from f to ρ" $\rho(r) = c_n\Psi^n$:

$$\begin{aligned}\int_0^{\epsilon_0} f(\epsilon)d\epsilon &= \frac{nc_n}{2\sqrt{2}\pi^2}\int_0^{\epsilon_0}\frac{\Psi^{n-1}}{\sqrt{\epsilon_0-\Psi}}d\Psi \qquad \text{Set } t \equiv \frac{\Psi}{\epsilon_0}\\ &= \frac{nc_n}{2\sqrt{2}\pi^2}\int_0^1 \frac{t^{n-1}\epsilon_0^n}{\sqrt{\epsilon_0}\sqrt{1-t}}dt\\ &= \frac{nc_n}{2\sqrt{2}\pi^2}\epsilon_0^{n-1/2}\beta\left(n,\frac{1}{2}\right) = \frac{nc_n}{2\sqrt{2}\pi^2}\frac{\Gamma(n)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)}\,\epsilon_0^{n-1/2}\end{aligned} \tag{541}$$

because $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. [Recall $\Gamma(n) = (n-1)!$]. Now differentiate to get

$$f(\epsilon_0) = \frac{nc_n}{2\sqrt{2}\pi^2}\left(n-\frac{1}{2}\right)\frac{(n-1)!\sqrt{\pi}}{\left(n-\frac{1}{2}\right)!}\,\epsilon_0^{n-3/2} = \frac{n!c_n}{(2\pi)^{3/2}\left(n-\frac{3}{2}\right)!}\epsilon_0^{n-3/2} = F\epsilon_0^{n-3/2}. \tag{542}$$

Therefore, we recover the DF used in the approach "from f to ρ", as we should.

Separable (Stäckel) potentials.

Separable (Stäckel) potentials are a spacial family of 3D potentials for which the equations of motion separate — and are explicitly known — in ellipsoidal coordinates (λ, μ, ν), defined as the roots of the equation:

$$\frac{x^2}{\tau+\alpha} + \frac{y^2}{\tau+\beta} + \frac{z^2}{\tau+\gamma} = 1, \tag{543}$$

where (x, y, z) are Cartesian coordinates and α, β and γ are constants determining the triaxial shape of the model. We adopt a convention $0 \le -\gamma \le \nu \le -\beta \le \mu \le -\alpha \le \lambda$.

All three integrals of motion have an analytic representation, as well as the density, potential and the DFs. Orbits in these potentials are combinations of oscillations and rotations in ellipsoidal coordinates. They are either tubes (along short and long axes) or boxes.

Whereas the separable potentials are not a very good fit to the observed galaxy density profiles (and are therefore of limited use in practice), they provide us with insight into the dynamics of triaxial systems: the orbits in other, physically more faithful integrable potentials, are generally of the same type as in separable potentials. For more on separable potentials, see the seminal paper by de Zeeuw (1985): `http://adsabs.harvard.edu/abs/1985MNRAS.216..273D`