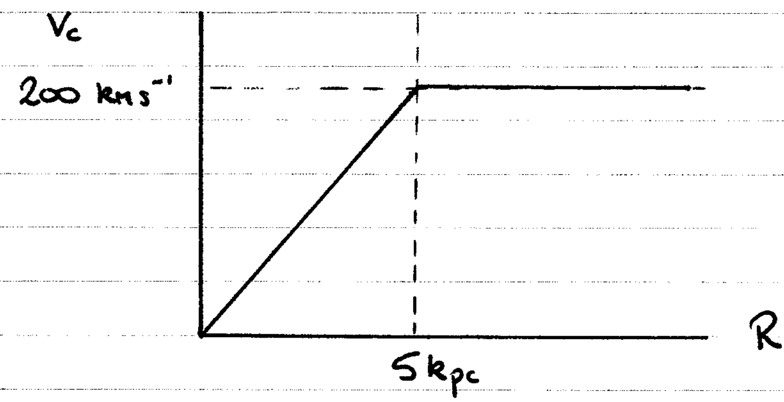
Problem Set # 1: Solutions

(1) Rotation curve:



5 points

(a) For spherically symmetric mass distribution:

$$\frac{V_c^2}{R} = \frac{GM(r)}{R^2}$$

$$\Rightarrow \quad M(R) = \frac{V_c^2}{G} R$$

- For $R < 5\ \mathrm{kpc}$

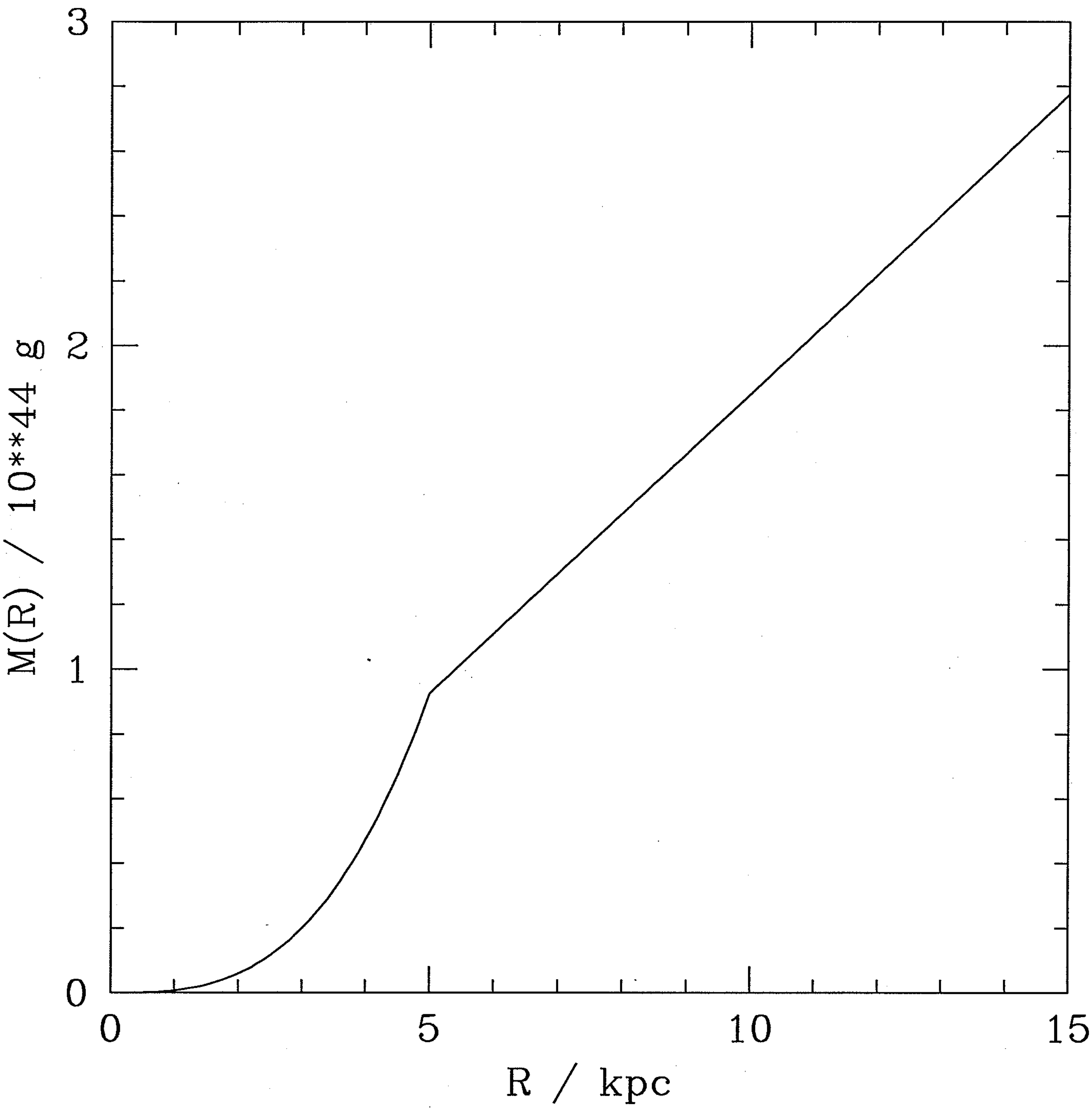
$$V_c = 200 \left(\frac{R}{5\,\mathrm{kpc}} \right) \ \mathrm{km\,s^{-1}}$$

$$V_c = 1.30 \times 10^{-15}\ R$$

$$M(R) = 7.40 \times 10^{41} \left(\frac{R}{\mathrm{kpc}} \right)^3 \ g$$

- For $R > 5\ \mathrm{kpc}$

$$M(R) = 1.85 \times 10^{43} \left(\frac{R}{\mathrm{kpc}} \right) \ g$$



M(R) / 10**44 g
R / kpc
0
1
2
3
0
5
10
15

5 points

(b) $$\frac{dM}{dR} = 4\pi R^2 \rho.$$

$$\Rightarrow \quad \rho_{DM} = \frac{1}{4\pi R^2} \frac{dM}{dR}$$

- For $R < 5$ kpc

$$\frac{dM}{dR} = 7.55 \times 10^{-23} R^2$$

$$\rho_{DM} = 6.0 \times 10^{-24} \text{ g cm}^{-3}$$

- For $R > 5$ kpc

$$\frac{dM}{dR} = 5.99 \times 10^{21}$$

$$\rho_{DM} = 2.0 \times 10^{-24} \left(\frac{R}{5\,\text{kpc}}\right)^{-2} \text{ g cm}^{-3}$$

[Note: this is not continuous @ 5 kpc]

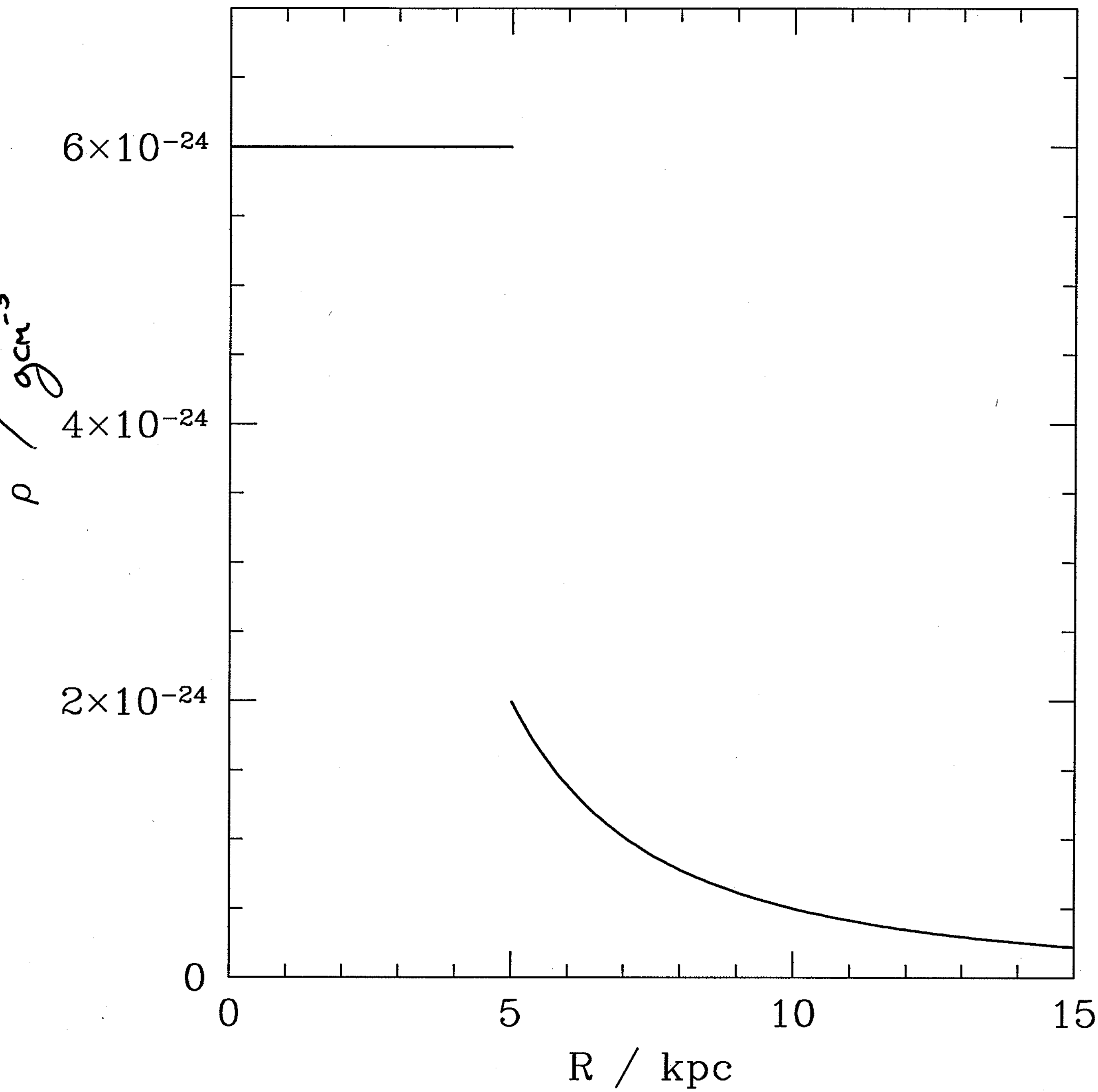
2 points

(c) At $R = 8$ kpc, $\rho_{DM} \simeq 7.8 \times 10^{-25}$ g cm^{-3}

In Solar System

$$M(< 1\,\text{AU}) = \tfrac{4}{3}\pi\,(1\,\text{AU})^3 \rho_{DM}$$

$$= 1.1 \times 10^{16} \text{ g}$$



6×10^{-24}
4×10^{-24}
2×10^{-24}
0
ρ / $g\,cm^{-3}$
0
5
10
15
R / kpc

8 points

(2) For sources of luminosity L distributed evenly in 3D space with number density n.

Flux @ distance d:

$$f = \frac{L}{4\pi d^2}$$

Volume within which sources will be visible to flux limit f_0

$$V = \frac{4}{3}\pi d_{max}^3$$

where $$d_{max} = \left(\frac{L}{4\pi f_0}\right)^{1/2}$$

$$\therefore \quad N(f > f_0) = nV \propto f_0^{-3/2}$$

For a disk the effective volume $V \propto d_{max}^2$

Rest of argument is unchanged, so

$$N(f > f_0) \propto f_0^{-1}$$

10 points

3 For $r < R$ we have

$$M(r) = \frac{V_c^2}{G} r$$

For $r > R$ we have

$$M(r) = \frac{V_c^2}{G} R \qquad \text{(a constant).}$$

Radial force: $F(r) = \dfrac{V_c^2}{r} \qquad r < R$

$$F(r) = \frac{V_c^2 R}{r^2} \qquad r > R$$

For $r < R$ gravitational potential:

$$\int_{\Phi}^{0} d\Phi = \int_{r}^{\infty} F(r)\, dr$$

$$= \int_{r}^{R} \frac{V_c^2}{r}\, dr + \int_{R}^{\infty} \frac{V_c^2 R}{r^2}\, dr$$

$$\Rightarrow \quad -\Phi = V_c^2 \ln\left(\frac{R}{r}\right) + V_c^2$$

Escape velocity: $\frac{1}{2} V_e^2 = -\Phi(r)$

$$\underline{\underline{V_e^2 = 2 V_c^2 \left(1 + \ln \frac{R}{r}\right)}}$$