Cluster modelling

Just as a self-gravitating collection of objects.

Collisions

Do we have to worry about collisions?

Globular clusters look densest, so obtain a rough estimate of collision timescale for them.

$\rho_0 \sim 8 \times 10^3$ M$_\odot$ pc^{-3}.

$M_* \sim 0.8$ M$_\odot$.

$\Rightarrow n_0 \sim 10^4$ pc^{-3} is the star number density.

We have $\sigma_r \sim 7$ km s^{-1} as the typical 1D speed of a star, so the 3D speed is $\sim \sqrt{3} \times \sigma_r$ $(= \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}) \sim 10$ km s^{-1}.

Since $M_* \propto R_*$ (see Fluids, or Stars, course notes), have $R_* \sim 0.8 R_\odot$.

For a collision, need the volume $\pi(2R_*)^2 \sigma t_{\rm coll}$ to contain one star, i.e.

$$n_0 = 1/\left(\pi(2R_*)^2 \sigma t_{\rm coll}\right) \tag{1}$$

or

$$t_{\rm coll} = 1/\left(4\pi R_*^2 \sigma n_0\right) \tag{2}$$

Putting in the numbers gives $t_{\rm coll} \sim 5 \times 10^{22}$ s $\sim 2 \times 10^{15}$ yr.

So direct collisions between stars are rare, but if you have $\sim 10^6$ stars then there is a collision every 2×10^9 years, so they <u>do</u> happen.

So, for now, ignore collisions, and we are left with stars orbiting in the potential from all the other stars in the system.

Model requirements

Have a gravitational potential well $\Phi(\mathbf{r})$, approximately smooth if the number of particles >> 1. Conventionally take $\Phi(\infty) = 0$.

Stars orbit in the potential well, with time per orbit (for a globular cluster) $\sim 2R_h/\sigma \sim 10^6$ years << age.

Stars give rise to $\Phi(\mathbf{r})$ by their mass, so for this potential in a steady state could average each star over its orbit to get $\rho(\mathbf{r})$.

The key problem is therefore self-consistently building a model which fills in the terms:

$$\Phi(\mathbf{r}) \rightarrow \text{ stellar orbits} \rightarrow \rho(\mathbf{r}) \rightarrow \Phi(\mathbf{r}) \tag{3}$$

Note that in most observed cases we only have $v_{\text{line of sight}}(R)$, so it is even harder to model real systems.

Self-consistent = orbits & stellar mass give ρ, which leads to Φ, which supports the orbits used to construct ρ.

Basics

Use:

Newtons laws of motion

Newtonian gravity

[General Relativity not needed, since $\bar{v} \sim 10 - 10^3$ km s^{-1} is $<< c = 3 \times 10^5$ km s^{-1} and $\frac{GM}{rc^2} \sim 2 \times 10^{-9}$ (globular cluster) $<< 1$.]

The gravitational force per unit mass acting on a body due to a mass M at the origin is

$$\mathbf{f} = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r} \tag{4}$$

We can write this in terms of a potential Φ, using

$$\begin{aligned} \nabla\left(\frac{1}{r}\right) &= \hat{\mathbf{r}}\frac{\partial}{\partial r}\left(\frac{1}{\mathbf{r}\cdot\mathbf{r}}\right)^{\frac{1}{2}} \quad (+0\times\hat{\theta}+0\times\hat{\phi}) \\ &= -\frac{1}{2}\hat{\mathbf{r}}\left(\frac{1}{\mathbf{r}\cdot\mathbf{r}}\right)^{\frac{3}{2}} 2\mathbf{r}\cdot\hat{\mathbf{r}} \\ &= -\frac{1}{r^2}\hat{\mathbf{r}} \end{aligned} \tag{5}$$

So

$$\mathbf{f} = -\nabla\Phi \tag{6}$$

where Φ is a scalar,

$$\Phi = \Phi(r) = -\frac{GM}{r} \tag{7}$$

Hence the potential due to a point mass M at $\mathbf{r} = \mathbf{r}_1$ is

$$\Phi(\mathbf{r}) = -\frac{GM}{|\mathbf{r} - \mathbf{r}_1|} \tag{8}$$

Orbits

Particle of constant mass m at position $\mathbf{r}$ subject to a force $\mathbf{F}$. Newton's law:

$$\frac{d}{dt}(m\dot{\mathbf{r}}) = \mathbf{F} \tag{9}$$

i.e.

$$m\ddot{\mathbf{r}} = \mathbf{F} \tag{10}$$

If $\mathbf{F}$ is due to a gravitational potential $\Phi(\mathbf{r})$, then

$$\mathbf{F} = m\mathbf{f} = -m\nabla\Phi \tag{11}$$

The angular momentum about the origin is $\mathbf{H} = \mathbf{r}\wedge(m\dot{\mathbf{r}})$.Then

$$\begin{aligned} \frac{d\mathbf{H}}{dt} &= \mathbf{r}\wedge(m\ddot{\mathbf{r}}) + m\dot{\mathbf{r}}\wedge\dot{\mathbf{r}} \\ &= \mathbf{r}\wedge\mathbf{F} \\ &\equiv \mathbf{G} \end{aligned} \tag{12}$$

where $\mathbf{G}$ is the torque about the origin.

The kinetic energy

$$T = \frac{1}{2}m\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} \tag{13}$$

$$\frac{dT}{dt} = m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = \mathbf{F}\cdot\dot{\mathbf{r}} \tag{14}$$

If $\mathbf{F} = -m\nabla\Phi$, then

$$\frac{dT}{dt} = -m\dot{\mathbf{r}}\cdot\nabla\Phi(\mathbf{r}) \tag{15}$$

But if Φ is independent of t, the rate of change of Φ along an orbit is

$$\frac{d}{dt}\Phi(\mathbf{r}) = \nabla\Phi\cdot\dot{\mathbf{r}}) \tag{16}$$

(from the chain rule).
Hence

$$\frac{dT}{dt} = -m\frac{d}{dt}\Phi(\mathbf{r}) \tag{17}$$

$$\Rightarrow m\frac{d}{dt}\left(\frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \Phi(\mathbf{r})\right) = 0 \tag{18}$$

$$\Rightarrow E = \frac{1}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \Phi(\mathbf{r}) \tag{19}$$

is constant for a given orbit.

Orbits in spherical potentials

$\Phi(\mathbf{r}) = \Phi(|\mathbf{r}|) = \Phi(r)$, so $\mathbf{f} = -\nabla\Phi = -\hat{\mathbf{r}}\frac{d\Phi}{dr}$.
The orbital angular momentum $\mathbf{H} = m\mathbf{r}\wedge\dot{\mathbf{r}}$, and

$$\frac{d\mathbf{H}}{dt} = \mathbf{r}\wedge m\mathbf{f} = -m\frac{d\Phi}{dr}\mathbf{r}\wedge\hat{\mathbf{r}} = 0. \tag{20}$$

So the angular momentum per unit mass $\mathbf{h} = \mathbf{H}/m = \mathbf{r}\wedge\dot{\mathbf{r}}$ is a constant vector, and is perpendicular to $\mathbf{r}$ and $\dot{\mathbf{r}}$

$\Rightarrow$ the particle stays in a plane through the origin which is perpendicular to $\mathbf{h}$.

[If you want this in more detail - $\mathbf{r} \perp \mathbf{h}$, $\mathbf{r} + \delta\mathbf{r} = \mathbf{r} + \dot{\mathbf{r}}\delta t \perp \mathbf{h}$ since both $\mathbf{r}$ and $\dot{\mathbf{r}} \perp \mathbf{h}$, so particle remains in the plane.]

Thus the problem becomes a two-dimensional one to calculate the orbit use 2-D cylindrical coordinates (R, ϕ, z) at $z = 0$, or spherical polars (r, θ, ϕ) with $\theta = \frac{\pi}{2}$.

So, in 2D, use (R, ϕ) and (r, ϕ) interchangeably..

Equation of motion in two dimensions

The equation of motion in two dimensions can be written in radial angular terms, using $\mathbf{r} = r\hat{\mathbf{r}} = r\hat{\mathbf{e}}_r + 0\hat{\mathbf{e}}_\phi$, so $\mathbf{r} =(r, 0)$.

We know that

$$\frac{d}{dt}\hat{\mathbf{e}}_r = \dot{\phi}\hat{\mathbf{e}}_\phi \tag{21}$$

and

$$\frac{d}{dt}\hat{\mathbf{e}}_\phi = -\dot{\phi}\hat{\mathbf{e}}_r \tag{22}$$

[To see this:

In time δt $\hat{\mathbf{e}}_r \to \hat{\mathbf{e}}_r + \delta\phi\hat{\mathbf{e}}_\phi$, and hence the first result above, and in the same time interval $\hat{\mathbf{e}}_\phi \to \hat{\mathbf{e}}_\phi - \delta\phi\hat{\mathbf{e}}_r$, which gives the second.]

Hence

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\phi}\hat{\mathbf{e}}_\phi \tag{23}$$

[or $\dot{\mathbf{r}} = \mathbf{v} =(\dot{r}, r\dot{\phi})$] and so

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{e}}_r + \dot{r}\dot{\phi}\hat{\mathbf{e}}_\phi + \dot{r}\dot{\phi}\hat{\mathbf{e}}_\phi + r\ddot{\phi}\hat{\mathbf{e}}_\phi - r\dot{\phi}^2\hat{\mathbf{e}}_r \\ &= (\ddot{r} - r\dot{\phi}^2)\hat{\mathbf{e}}_r + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\mathbf{e}}_\phi \\ &= \mathbf{a} = [\ddot{r} - r\dot{\phi}^2, \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)] \end{aligned} \tag{24}$$

In general $\mathbf{f} =(f_r, f_\phi)$, and then $f_r = \ddot{r} - r\dot{\phi}^2$, where the second term is the centrifugal force, since we are in a rotating frame, and the torque $rf_\phi = \frac{d}{dt}\left(r^2\dot{\phi}\right)$ $(= \mathbf{r}\wedge\mathbf{f})$.

In a spherical potential $f_\phi = 0$, so $r^2\dot{\phi}$ is constant.

Path of the orbit

To determine the shape of the orbit we need to remove t from the equations and find $r(\phi)$. It is simplest to set $u = 1/r$, and then from $r^2\dot{\phi} = h$ obtain

$$\dot{\phi} = hu^2 \tag{25}$$

Then

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{d\phi}\dot{\phi} = -h\frac{du}{d\phi} \tag{26}$$

and

$$\ddot{r} = -h\frac{d^2u}{d\phi^2}\dot{\phi} = -h^2u^2\frac{d^2u}{d\phi^2}. \tag{27}$$

So the radial equation of motion

$$\ddot{r} - r\dot{\phi}^2 = f_r$$

becomes

$$-h^2u^2\frac{d^2u}{d\phi^2} - \frac{1}{u}h^2u^4 = f_r \tag{28}$$

$$\Rightarrow \frac{d^2u}{d\phi^2} + u = -\frac{f_r}{h^2u^2} \tag{29}$$

Since f_r is just a function of r (or u) this is an equation for $u(\phi)$, i.e. $r(\phi)$ - the path of the orbit. Note that it does not give $r(t)$, or $\phi(t)$ - you need one of the other equations for those.
If we take $f_r = -\frac{GM}{r^2} = -GMu^2$, then

$$\frac{d^2u}{d\phi^2} + u = GM/h^2 \tag{30}$$

(which is something you will have seen in the Relativity course).
The solution to this equation is

$$\frac{\ell}{r} = \ell u = 1 + e\cos(\phi - \phi_0) \tag{31}$$

which you can verify simply by putting it in the differential equation. Then

$$-\frac{e\cos(\phi-\phi_0)}{\ell} + \frac{1 + e\cos(\phi-\phi_0)}{\ell} = \frac{GM}{h^2}$$

so $\ell = h^2/GM$ and e and ϕ_0 are constants of integration.

Note that if $e < 1$ then $1/r$ is never zero, so r is bounded in the range $\frac{\ell}{1+e} < r < \frac{\ell}{1-e}$. Also, in all cases the orbit is symmetric about $\phi = \phi_0$, so we take $\phi_0 = 0$ as defining the reference line for the angle ϕ. ℓ is the distance from the origin for $\phi = \pm\frac{\pi}{2}$ (with ϕ measured relative to ϕ_0).

We can use different parameters. Knowing that the point of closest approach (perihelion for a planet in orbit around the sun, periastron for something about a star) is at $\ell/(1+e)$ when $\phi = 0$ and the aphelion (or whatever) is at $\ell/(1-e)$ when $\phi = \pi$, we can set the distance between these two points (= major axis of the orbit)=$2a$. Then

$$\frac{\ell}{1+e} + \frac{\ell}{1-e} = 2a \Rightarrow \ell(1-e) + \ell(1+e) = 2a(1-e^2) \tag{32}$$

$$\Rightarrow \ell = a(1-e^2) \tag{33}$$

$\Rightarrow r_P = a(1-e)$ is the perihelion distance from the gravitating mass at the origin, and $r_a = a(1+e)$ is the aphelion distance. The distance of the sun from the midpoint is ae, and the angular momentum $h^2 = GM\ell = GMa(1-e^2)$. We can transform to Cartesian (x, y) setting $x = r\cos\phi + ae$ and $y = r\sin\phi$ so the origin is at the midpoint of the major axis. Then fairly uninstructive algebra gives the standard Cartesian equation for an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{34}$$

where b is the length of the minor axis, $b^2 = a^2(1-e^2)$.

Energy per unit mass

The energy per unit mass

$$E = \frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \Phi(\mathbf{r}) = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r} \tag{35}$$

This is constant along the orbit, so we can evaluate it anywhere convenient - e.g. at perihelion where $\dot{r} = 0$. Then $\dot{\phi} = \frac{h}{r_P^2}$ and so

$$\begin{aligned} E &= \frac{1}{2}\frac{GMa(1-e^2)}{a^2(1-e)^2} - \frac{GM}{a(1-e)} \\ &= \frac{GM}{a}\left[\frac{1}{2}\left(\frac{1+e}{1-e}\right) - \frac{1}{1-e}\right] \\ &= -\frac{GM}{2a} \end{aligned} \tag{36}$$

This is < 0 for a bound orbit, and is depends only on the semi-major axis a (and not e).

Kepler's Laws

... deduced from observations, and explained by Newtonian theory of gravity.

1 Orbits are ellipses with the sun at a focus.

2 Planets sweep out equal areas in equal time

$$\delta A = \frac{1}{2}r^2\delta\phi \;\; [= \frac{1}{2}r(r\delta\phi)] \tag{37}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \frac{h}{2} = \text{constant} \tag{38}$$

$\Rightarrow$ Kepler's second law is a consequence of a central force, since this is why h is a constant.

3 (Period)$^2 \propto$ (size of orbit)^3

In one period T, the area swept out is $A = \frac{1}{2}hT = \left(\int_0^\infty \frac{dA}{dt}dt\right)$
But $A =$ area of ellipse $= \pi ab = \pi a^2\sqrt{1-e^2}$
[

$$A = \int_0^{2\pi} d\phi \int_0^r r dr$$
$$= \int_0^{2\pi} \frac{1}{2}r^2 d\phi$$
$$= \frac{\ell^2}{2}\int_0^{2\pi} \frac{d\phi}{(1+e\cos\phi)^2}$$

Have

$$\int_0^\pi \frac{dx}{(a+b\cos x)^2} = \frac{\pi}{a^2-b^2}\frac{a}{\sqrt{a^2-b^2}}$$

so

$$A = 2\frac{\ell^2}{2}\frac{\pi}{1-e^2}\frac{1}{\sqrt{1-e^2}}$$

Since $\ell = a(1-e^2)$ this implies

$$A = \pi a^2\sqrt{1-e^2}$$

and since $b = a\sqrt{1-e^2}$,

$$A = \pi ab$$

]
Therefore

$$\begin{aligned} T &= \frac{2\pi a^2\sqrt{1-e^2}}{h} \\ &= \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{GMa(1-e^2)}} \\ [\text{since } h^2 &= GMa(1-e^2)\ ] \\ T &= 2\pi\sqrt{\frac{a^3}{GM}} \\ \Rightarrow T^2 &\propto a^3 \end{aligned} \tag{39}$$

where in this case M is the mass of the sun.
Note: Since $E = -\frac{GM}{2a}$, the period $T = \frac{2\pi GM}{(-2E)^{\frac{3}{2}}}$.

Unbound orbits

$\frac{\ell}{r} = 1 + e\cos\phi$ with $e \geq 1$.
If $e > 1$ then $1 + e\cos\phi = 0$ has solutions ϕ_∞ where $r = \infty$.

$$\rightarrow \cos\phi_\infty = -1/e$$

Then $-\phi_\infty \leq \phi \leq \phi_\infty$, and, since $\cos\phi_\infty$ is negative, $\frac{\pi}{2} < \phi_\infty < \pi$.

The orbit is a hyperbola. If $e = 1$ then the particle just gets to infinity at $\phi = \pm\pi$ - it is a parabola.
Energies for these unbound orbits:

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{h^2}{r^2} - \frac{GM}{r}$$

as $r \rightarrow \infty$ $E \rightarrow \frac{1}{2}\dot{r}^2$.
Recall

$$\frac{\ell}{r} = 1 + e\cos\phi$$

$\frac{d}{dt}$ of this $\Rightarrow$

$$-\frac{\ell}{r^2}\dot{r} = -e\sin\phi\ \dot{\phi}$$

and since $h = r^2\dot{\phi}$

$$\dot{r} = \frac{eh}{\ell}\sin\phi$$

As $r \rightarrow \infty$ $\cos\phi \rightarrow -1/e$

$$E \rightarrow \frac{1}{2}\dot{r}^2 = \frac{1}{2}\frac{e^2h^2}{\ell^2}\left(1 - \frac{1}{e^2}\right) = \frac{GM}{2\ell}(e^2 - 1)$$

(recalling that $h^2 = GM\ell$) Thus $E > 0$ if $e > 1$ and for parabolic orbits $(e = 1)$ $E = 0$.

Escape velocity

We have seen that in a fixed potential $\Phi(\mathbf{r})$ a particle has constant energy $E = \frac{1}{2}\dot{\mathbf{r}}^2 + \Phi(\mathbf{r})$ along an orbit. If we adopt the usual convention and take $\Phi(\mathbf{r}) \to 0$ as $|\mathbf{r}| \to \infty$, then if at some point $\mathbf{r}_0$ the particle has velocity $\mathbf{v}_0$ such that

$$\frac{1}{2}\mathbf{v}_0^2 + \Phi(\mathbf{r}_0) > 0$$

then it is able to reach infinity. So at each point $\mathbf{r}_0$ we can define an escape velocity $v_{\rm esc}$ such that

$$v_{\rm esc} = \sqrt{-2\Phi(\mathbf{r}_0)}$$

The escape velocity from the sun

$$v_{\rm esc} = \left(\frac{2GM_\odot}{r_0}\right)^{\frac{1}{2}}$$

$$= 42.2 \left(\frac{r_0}{\text{a.u.}}\right)^{-\frac{1}{2}} \text{ km s}^{-1}$$

Note: The circular velocity $v_{\rm circ}$ is such that $-r\dot{\phi}^2 = -\frac{GM}{r^2}$

$$r\dot{\phi} = v_{\rm circ} = \sqrt{\frac{GM_\odot}{r_0}} = 29.8 \left(\frac{r_0}{\text{a.u.}}\right)^{-\frac{1}{2}} \text{ km s}^{-1}$$

$(= 2\pi$ a.u./yr).
$v_{\rm esc} = \sqrt{2}v_{\rm circ}$ for a point mass source of the gravitational potential.