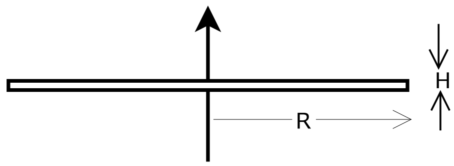
Potential due to thin disk



$H << R$

Could use above and write down potential as sum over rings.

But method does not work at $r = a$.

Instead use cylindrical polar, (R, ϕ, z), coordinates

Expect $\Phi \equiv \Phi(R, z)$ and $\Phi(R, z) = \Phi(R, -z)$ by symmetry.

Outside disk $\nabla^2 \Phi = 0$.

$$\Rightarrow \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

We solve this by separation of variables, letting

$$\Phi(R, z) = J(R)Z(z)$$

$$\Rightarrow \quad Z(z)\frac{1}{R}\frac{d}{dR}\left(R\frac{dJ(R)}{dR}\right) + J(R)\frac{d^2}{dz^2}Z(z) = 0$$

$$\Rightarrow \quad \underbrace{\frac{1}{JR}\frac{d}{dR}\left(R\frac{dJ}{dR}\right)}_{\text{function of } R} = - \underbrace{\frac{1}{Z}\frac{d^2 Z}{dz^2}}_{\text{function of } z} = -k^2 \text{ , say}$$

$$\Rightarrow \quad \frac{d^2 Z}{dz^2} - k^2 Z = 0 \qquad (9\text{-}1)$$

so $\quad Z = A\exp(kz) + B\exp(-kz)$

$$\text{and} \quad \frac{1}{R}\frac{d}{dR}\left(R\frac{dJ}{dR}\right) + k^2 J(R) = 0 \qquad (9\text{-}2)$$

We would quite like $\Phi(R, \infty)$ and $\Phi(R, -\infty)$ to be zero, so

$$Z(z) = A \exp(-k|z|)$$

is the appropriate solution for $Z(z)$.

The R equation (9-2) is the defining equation for a Bessel function. These are the analogues of sines and cosines now for cylindrical as opposed to linear problems (e.g. drum beats).

So while $$\frac{d^2y}{dz^2} + d^2y = 0$$ has solutions $\sin(kz)$, $\cos(kz)$ (9-3)

similarly $$\frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) + k^2y = 0$$ has solutions $J_0(ks)$, $Y_0(ks)$ (9-4)

which you can look up in e.g. Abramowitz & Stegun "Handbook of Mathematical Functions".

Examples are given on the next page.

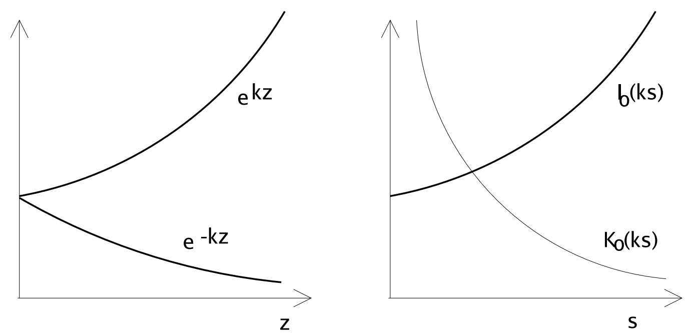
Note that as $x \to 0$ $J_0(x) \to 1$ and $Y_0(x) \to -\infty$.
More generally the equation

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) + \left(k^2 - \frac{\nu}{s^2}\right)y = 0$$

has solutions $J_\nu(ks), Y_\nu(ks)$, so we get while a whole family of Bessel functions characterized by the index ν.
Also there are "modified" Bessel functions where $k \to ik$

$$\frac{d^2y}{dz^2} - k^2 y = 0$$

has solutions $\sin(ikz)$, $\cos(ikz)$ or $\exp(kz)$



Similarly $\quad \frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) - k^2 y = 0 \to \quad I_0(ks)$

$K_0(ks)$

and $\quad \frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) - \left(k^2 + \frac{\nu^2}{s^2}\right)y = 0 \to \quad I_\nu(ks)$, $K_\nu(ks)$

see Abramowitz + Stegun "Handbook of Mathematical functions"
And we can take this even further. By analogy with Fourier transforms where $\sin, \cos \to$ form the basis, we have $J, Y \to$ Hankel transforms.
Given a function $g(r)$, then the Hankel transform of g is

$$\tilde{g}(k) = \int_0^\infty g(r) J_\nu(kr) r dr$$

and the inverse transform is:

$$g(r) = \int_0^\infty \tilde{g}(k) J_\nu(dr) k dk$$

$\Big[$ look these up in books of Hankel transforms! $\Big]$

Returning to the axisymmetric plane distribution, we have
(9-1) $\Rightarrow Z(z) = \exp(-k|z|)$
(9-2) $\Rightarrow J(R) = J_0(kr)$
choose J to get Φ finite at $R = 0$

Let $k > 0$ then

$$\Rightarrow \quad \Phi_k(R, Z) = \begin{array}{ll} Ce^{-kz} J_0(kR) & z > 0 \\ \\ Ce^{kz} J_0(kR) & z < 0 \end{array}$$

This is true $\forall\ k > 0$, but a specific k for each Φ_k.
General potential $\rightarrow \sum_k \Phi_k$

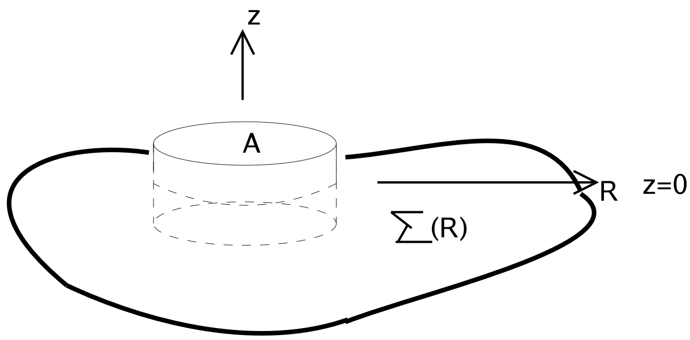
$$\Rightarrow \Phi(R, z) = \int_0^\infty f(k) e^{-k|z|} J_0(kR) dk \qquad (9\text{-}5)$$

Here $f(k)$ is a weighting function, corresponding to the C values in the sum. So what we need to do for a particular mass distribution is find $f(k)$.

If we are going to relate it to a mass distribution, the next thing we should do is look at the $z = 0$ plane, i.e. the region we have neglected so far since we have taken $\nabla^2 \Phi = 0$ and so considered regions outside the plane.

Note that Φ_k is continuous across $z = 0$ but $\nabla \Phi_k$ is not due to $|z|$ dependence.That is where the mass is, so that is not a surprise.

$\Rightarrow \nabla^2 \Phi_k = 0$ except at $z = 0$ and $\Phi_k \rightarrow 0$ as $z, R \rightarrow \infty \Rightarrow$ satisfies conditions for potential from an isolated mass distribution. Still need to link with ρ (or $\Sigma(R)$) in the plane.



Use Gauss' Theorem ($\equiv$ Poisson's equation plus divergence theorem) to determine Σ in the $z = 0$ plane.

Over the cylinder

$$\iiint 4\pi G\rho dV = \iiint \nabla^2\Phi dV = \iiint \nabla \cdot (\nabla\Phi)\, dV$$
$$= \iint \nabla\Phi \cdot \hat{\mathbf{n}} d^2\mathbf{S}$$

Consider the limit in which the cylinder height $\to 0$. Then if A is the area of an end of the cylinder

$$LHS = 4\pi G\Sigma A$$
$$RHS = \left(\left[\frac{\partial\Phi}{\partial z}\right]_{z=0+} - \left[\frac{\partial\Phi}{\partial z}\right]_{z=0-} \right) \times A$$
$$\text{Equating these } \Rightarrow \left[\frac{\partial\Phi}{\partial z}\right]_{0-}^{0+} = 4\pi G\Sigma(R)$$

$$\begin{aligned}
LHS &= -\int_0^\infty kf(k)e^{-k0+}J_0(kR)dk - \int_0^\infty kf(k)^{z=0}e^{-k0-}J_0(kR)dk \\
&= -\int_0^\infty kf(k)J_0(kR)dk - \int_0^\infty kf(k)J_0(kR)dk \\
&= -2\int_0^\infty kf(k)J_0(kR)dk \\
\Rightarrow \quad \Sigma(R) &= -\frac{1}{2\pi G}\int_0^\infty f(k)J_0(kR)kdk
\end{aligned}$$

Hence determine $f(k)$ [and hence Φ] from inverse Hankel transform

$$f(k) = -2\pi G \int_0^\infty \Sigma(R) J_0(kR) r dR$$

Thus the process for dtermining Φ from ρ in this case is $\Sigma \to f \to \Phi$.

Note: For determining the circular velocity need $\frac{\partial \Phi}{\partial R}$, whch becomes $\frac{dJ_0(x)}{dx}$, and for Bessel function J_0 have $\frac{dJ}{dx} J_0(x) = -J_1(x)$ [Example].

This has been a bit longwinded, but the steps are clear. They are:

Summary of derivation of Φ for thin axisymmetric disk

1. $\nabla^2 \Phi = 0$ outside disk. Solve by separation of variables.
2. Solutions of form $\Phi_k(R, z) = Ce^{-k|z|} J_0(kR) \; \forall \; k > 0$
3. $\Phi_k \to 0$ as $R, z \to \infty$ **and** satisfies $\nabla^2 \Phi = 0$
   $\Rightarrow$ is potential of an isolated density distribution
4. General Φ can be written as

   $$\Phi(R, z) = \int_0^\infty \Phi_k(R, z) f(k) dk$$

   where $f(k)$ is an appropriate weight function.
5. Use Gauss' theorem to determine

   $$\Sigma(R) = -\frac{1}{2\pi G} \int_0^\infty f(k) J_0(kR) k dk$$

6. Hence

   $$f(k) = -2\pi G \int_0^\infty \Sigma(R) J_0(kR) R dR$$

So given Σ, use item (6) to determine $f(k)$, and then (5) to obtain Φ.

The circular velocity in the plane of a plane distribution of matter is given by

$$\frac{v_C^2(R)}{R} = \left. \frac{\partial \Phi}{\partial R} \right|_{z=0}$$

and we have

$$x = kR \qquad \frac{d}{dR}J_0(kR) = k\frac{d}{dx}J_0 x = -kJ_1(kR)$$

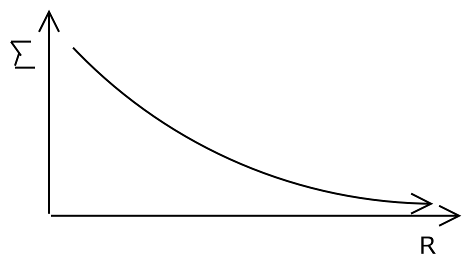
Then since we have equation (9-5) [$\Phi(R,z) = \int_0^\infty f(k)e^{-k|z|}J_0(kR)dk$] then

$$\frac{v_C^2(R)}{R} = -\int_0^\infty f(k)J_1(kR)kdk$$

Examples

(a) Mestel disk

A Mestel disk has the surface density distribution $\quad \Sigma(R) = \frac{\Sigma_0 R_0}{R}$



Thus

$$\begin{aligned} M(<R) = \int_0^R 2\pi\Sigma(R')R'dR' &= 2\pi R_0\Sigma_0\int_0^R dR' \\ &= 2\pi\Sigma_0 R_0 R \\ &\to \infty \text{ as } R \to \infty \end{aligned}$$

$$\begin{aligned} f(k) &= -2\pi G\Sigma_0 R_0 \int_0^\infty J_0(kR)dR \\ &= -\frac{2\pi G\Sigma_0 R_0}{k} \end{aligned}$$

$$\left[\begin{array}{l} \text{From Gradshteyn and Ryzkik 6.511.1} \\ \\ \displaystyle\int_0^\infty J_\nu(bx)dx = \frac{1}{b} \quad \begin{array}{c} Re(\nu) > -1 \\ b > 0 \end{array} \end{array} \right]$$

$$\begin{aligned} \Rightarrow \Phi(R,z) &= -2\pi G \Sigma_0 R_0 \int_0^\infty e^{-k|z|} \frac{J_0(kR)}{k} dk \\ \text{and} \quad \frac{v_c^2(R)}{R} &= 2\pi G \Sigma_0 R_0 \int_0^\infty J_1(kR) dk \\ \Rightarrow v_c^2(R) &= 2\pi G \Sigma_0 R_0 = \text{ const} \end{aligned}$$

Note that

$$v_c^2(R) = \frac{GM(R)}{R}$$

exactly in this case even though distribution is a disk, not spherical.
More generally, find $v_c^2 \equiv \frac{GM(R)}{R}$ to within 10% [reasonable accuracy] for most smooth Σ distributions. (see figure on next page)

Conclude that measurement of $v_c(R)$ is a good measure of mass **inside** R.

Exponential Disk

Here

$$\Sigma(R) = \Sigma_0 \ \exp\left[-R/Rd\right] \tag{9-6}$$

This has finite mass

$$\begin{aligned} M &= \int_0^\infty 2\pi\Sigma_0 \ \exp\left[-R/Rd\right] RdR \\ &= 2\pi\Sigma_0 R_d^2 \underbrace{\int_0^\infty e^{-x} x dx}_{=1} \\ &= 2\pi\Sigma_0 R_d^2 \end{aligned}$$

Then

$$f(k) = -2\pi G\Sigma_0 \int_0^\infty e^{-R/Rd} J_0(kR) RdR$$

$$\left[\ \textit{Gradshteyn} + \textit{Ryzhik}: \quad \int_0^\infty e^{-\alpha x} J_0\left(\beta x\right) x dx = \frac{\alpha}{\left[\beta^2 + \alpha^2\right]^{3/2}}\ \right]$$

[Actually they have something (6.632.2) which requires a little work:

$$\int_0^\infty J_\nu(\beta x) x^{\nu+1} dx = \frac{2\alpha (2\beta)^\nu \Gamma(\nu + \frac{3}{2})}{\sqrt{\pi}\left(\alpha^2 + \beta^2\right)^{\nu + \frac{3}{2}}}$$

and you need to put in $\nu = 0$, $\Gamma(3/2) = \sqrt{\pi}/2$.
Then put $\alpha = 1/R_d$ and $\beta = k$.]

$$f(k) = -\frac{2\pi G\Sigma_0 R_d^2}{\left[1 + (kR_d)^2\right]^{3/2}}$$

Hence

$$\Phi(R, z) = -2\pi G\Sigma_0 R_d^2 \int_0^\infty \frac{J_0(kR) e^{-k|z|}}{\left(1 + (kR_d)^2\right)^{3/2}} dk$$

Use

$$\int_0^\infty \frac{J\nu(xy) dx}{(x^2 + a^2)^{1/2}} = I_{\nu/2}\left(\frac{1}{2} ay\right) K_{\nu/2}\left(\frac{1}{2} ay\right)$$

You can do this with help from Gradshteyn + Ryzhik again, using (6.552.1)

$$\int_0^\infty \frac{J_\nu(xy)\ dx}{\sqrt{x^2+a^2}} = I_{\nu/2}(\frac{1}{2}ay)K_{\nu/2}(\frac{1}{2}ay)$$

for $Re(a) > 0$, $y > 0$, $Re(\nu) > -1$,
and $I_0'(z) = I_1(z)$, $K_0'(z) = -K_1(z)$

$\frac{d}{da}$ of this gives

$$-a\int_0^\infty \frac{J_\nu(xy)\ dx}{x^2+a^{2^{3/2}}} = \frac{y}{2}I_{\nu/2}(\frac{1}{2}ay)K_{\nu/2}'(\frac{1}{2}ay) + \frac{y}{2}I_{\nu/2}'(\frac{1}{2}ay)K_{\nu/2}(\frac{1}{2}ay)$$

so for $\nu = 0$

$$-a\int_0^\infty \frac{J_0(xy)\ dx}{x^2+a^{2^{3/2}}} = -\frac{y}{2}I_0(\frac{1}{2}ay)K_1(\frac{1}{2}ay) + \frac{y}{2}I_1(\frac{1}{2}ay)K_0(\frac{1}{2}ay)$$

or

$$\int_0^\infty \frac{J_\nu(xy)\ dx}{x^2+a^{2^{3/2}}} = \frac{y}{2a}\left[I_0(\frac{1}{2}ay)K_1(\frac{1}{2}ay) - I_1(\frac{1}{2}ay)K_0(\frac{1}{2}ay)\right]$$

Then with $x = k$, $y = R$ and $a = 1/R_d$ this becomes

$$\int_0^\infty \frac{J_\nu(kR)\ dk}{1+(kRd)^{2^{3/2}}} = \frac{R}{2R_d^2}\left[I_0(\frac{R}{2R_d})K_1(\frac{R}{2R_d}) - I_1(\frac{R}{2R_d})K_0(\frac{R}{2R_d})\right]$$

Also you find for the circular velocity (with $y = R/2R_d$)

$$v_C^2 = R\frac{\partial\Phi}{\partial R} = 4\pi\Sigma_0 R_d y^2[I_0K_0 - I_1K_1]$$

which is helpfully left as an example...