Part A : Gravity

1 Recap of Newtonian Dynamics

Consider a particle of mass m and velocity $\mathbf{v}$. Momentum of the particle is defined as $\mathbf{p} = m\mathbf{v}$. Newton's Second Law (N2) states that, if particle is acted upon by force $\mathbf{F}$, then

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m\mathbf{a}, \tag{1}$$

where $\mathbf{a} = d\mathbf{v}/dt$ and m =constant.

If $\mathbf{F} = 0$ then $\mathbf{p}$ =constant, giving us Newton's First Law (N1).

Newton's Third Law (N3) states that, if body-A exerts force $\mathbf{F}$ on body-B, then body-B exerts a force $-\mathbf{F}$ on body-A.

Consider a system of bodies with masses m_i, velocities $\mathbf{v}_i$ and momentum $\mathbf{p}_i$ ($i = 1, ...N$). Total momentum is

$$\mathbf{p}_{\text{tot}} = \sum_{i=1}^{N} \mathbf{p}_i. \tag{2}$$

We can show (ON BOARD) that the total momentum is constant if the system is not subjected to any external forces.

In class, we will talk about

- Meaning of an isolated system
- Some generalized definitions of momentum

2 Newtonian Gravity

Newton's Law of Universal Gravitation is : Suppose that mass M_1 is at the origin and mass M_2 is at position r. Then the gravitational force of M_1 on M_2 is

$$F_{21} = -\frac{GM_1M_2}{|\mathbf{r}|^2}\hat{\mathbf{r}}. \tag{3}$$

More generally, we can define the gravitational field at each point in space $\mathbf{g}(\mathbf{r}$ such that the gravitational force felt by a particle of mass m is

$$\mathbf{F} = m\mathbf{g}. \tag{4}$$

Newton's Law of Gravitation now gives us the gravitational field at point $\mathbf{r}$ from a mass M at point $\mathbf{r}'$,

$$\mathbf{g} = -\frac{GM}{|\mathbf{r} - \mathbf{r}'|^3}(\mathbf{r} - \mathbf{r}'). \tag{5}$$

For a distributed mass with density $\rho(\mathbf{r})$ filling volume V, we can just treat it as a set of point masses to write,

$$\mathbf{g} = -G \int_V \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3\mathbf{r}'. \tag{6}$$

From this, we can prove (ON BOARD) Gauss's Law of Gravity,

$$\nabla \cdot \mathbf{g} = -4\pi G \rho(\mathbf{r}). \tag{7}$$

Introducing the gravitational potential Φ such that $\mathbf{g} = \nabla\Phi$, Gauss's Law can be written as

$$\nabla^2 \Phi = 4\pi G \rho, \tag{8}$$

a form known as Poisson's equation.

In class, we will talk about

- Geometric meaning of Gauss's Law
- Application to "Newton's Shell Theorem"
- Analogy to electrostatics

3 The One-Body Problem

3.1 Problem setup and basic equation of motion

Consider an object of mass m moving within the gravitational field of a much more massive object with mass M (i.e. $M \gg m$). Otherwise, the system is isolated. Furthermore, we assume that the only forces acting on m are gravitational. Place M at the origin. Then the equation of motion is

$$\ddot{\mathbf{r}} = -\frac{GM}{r^2}\hat{\mathbf{r}}, \tag{9}$$

where we write $r \equiv |\mathbf{r}|$.

3.2 Conservation Laws

From this equation of motion, we can quickly derive (ON BOARD) two important conservation laws, the conservation of angular momentum:

$$\text{Ł} \equiv \mathbf{r} \times \mathbf{p} = \text{constant}, \tag{10}$$

and the conservation of total energy

$$E = \frac{1}{2}mv^2 - \frac{GM}{r} = \text{constant}. \tag{11}$$

3.3 Solving the equation of motion

To make more progress, we need to actually solve the equation of motion. Must choice some specific coordinate system so we can describe the components of $\mathbf{r}$ and $\dot{\mathbf{r}}$. The most natural choice is to work in cylindrical polar coordinates (R, ϕ, z) oriented such that the angular momentum $\mathbf{L}$ points along the z-axis and the motion is confined into the plane $z = 0$. We can write the basic vectors of the polar coordinates $(\hat{\mathbf{R}}, \hat{\phi})$ in terms of the usual Cartesian basis vectors $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$:

$$\begin{aligned} \hat{\mathbf{R}} &= \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} &(12)\\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{x}}. &(13) \end{aligned}$$

Using this, and starting with $\mathbf{r} = R\hat{\mathbf{R}}$, we can show (ON BOARD) that

$$\dot{\mathbf{r}} = \dot{R}\hat{\mathbf{R}} + R\dot{\phi}\hat{\boldsymbol{\phi}}. \tag{14}$$

We can now translate the conservation laws into polar coordinates. We find (ON BOARD) that $\mathbf{L} = L_z\hat{\mathbf{z}}$ with

$$L_z = mR^2\dot{\phi}. \tag{15}$$

We define the specific angular momentum $\Lambda \equiv L_z/m$. Hence

$$\Lambda = R^2\dot{\phi}. \tag{16}$$

We also define the specific energy $\epsilon \equiv E/m$ and can show (ON BOARD) that

$$\epsilon = \frac{1}{2}\dot{R}^2 + \frac{1}{2}\frac{\Lambda^2}{R^2} - \frac{GM}{R} \tag{17}$$

Equation (17) is an ordinary differential equation (ODE) for R in terms of time t. To solve, it turns out to be convenient to make a change of variable to $u \equiv 1/R$. Then we can show (ON BOARD) that

$$\dot{R} = -\lambda\frac{du}{d\phi}, \tag{18}$$

and by substituting this into eqn. (17) and differentiating w.r.t. ϕ we find that

$$\frac{du}{d\phi}\left[\Lambda^2\left(\frac{d^2u}{d\phi^2} + u\right) - GM\right] = 0. \tag{19}$$

There are two ways that this equation can be satisfied.

Case A : $du/d\phi = 0$, i.e. u =constant. This is circular motion! We can show (ON BOARD) that, in this case, we have $\Lambda = \sqrt{GMR}$, $\dot{\phi} = \sqrt{GM/R^3}$ and $\epsilon = -GM/2R$.

Case B : : $d^2u/d\phi^2 + u - GM/\Lambda^2 = 0$. This is a second-order linear ODE with general solution

$$u = \frac{GM}{\Lambda^2} + c\,\cos(\phi - \phi_0), \tag{20}$$

where c and ϕ_0 are (integration) constants. Relating c to the conserved quantities (Λ and ϵ) and re-writing back in terms of R gives us the solution for $R(\phi)$ that we seek:

$$\frac{1}{R} = \frac{GM}{\Lambda^2}\left[1 + c\cos(\phi - \phi_0)\right], \text{ with } c = \left[1 + \frac{2\Lambda^2\epsilon}{(GM)^2}\right]^{1/2}. \tag{21}$$

Can immediately see that solutions only exist provided that $\epsilon \geq -(GM)^2/2\Lambda^2$. For specific energies that are more negative than this, there are no solutions (at least in the real domain!). In fact, eqn. (21) encompasses both circular and non-circular motion (i.e. includes Case-A as a special case when $c = 0$).

Let's examine some general properties of this solution (eqn. 21). For $\epsilon < 0$, R is finite for all ($\forall$) values of ϕ. Furthermore, the solution is clearly periodic in ϕ with period 2π. Hence, this solution describes an orbit which closes on itself. The maximum value of R is when $\cos(\phi - \phi_0) = -1$, giving

$$\frac{1}{R_{\rm max}} = \frac{GM}{\Lambda^2}\left[1 - \left(1 + \frac{2\Lambda\epsilon}{(GM^2)}\right)^{1/2}\right]. \tag{22}$$

Similarly, minimum radius is when $\cos(\phi - \phi_0) = +1$ and

$$\frac{1}{R_{\rm min}} = \frac{GM}{\Lambda^2}\left[1 + \left(1 + \frac{2\Lambda\epsilon}{(GM^2)}\right)^{1/2}\right]. \tag{23}$$

A little algebra (ON BOARD) reveals that

$$R_{\rm min} + R_{\rm max} = \frac{GM}{|\epsilon|}. \tag{24}$$

So the total "length" of the orbit is purely a function of energy.

For $\epsilon > 0$, we have that $c > 1$ and hence eqn. (21) would tell us that $1/R$ hits zero. This means that $R \to \infty$ as $\phi - \phi_0 \to \pm\phi_m$ where $\phi_m = \cos^{-1}(-1/c)$. So, the particle comes in from infinity from a direction $\phi = \phi_0 + \phi_m$, swings by the origin (reaching closest approach at $\phi = \phi_0$ and then heads back out to infinity along $\phi = \phi_0 - \phi_m$.

For $\epsilon = 0$, we have $R \to \infty$ as $\phi - \phi_0 \to \pi$. So, particle comes in from infinity along a direction $\phi = \phi_0 + \pi$, swings by the origin (reaching closest approach at $\phi = \phi_0$ and then heads back out to infinity along $\phi = \phi_0 - \pi$.

3.4 Properties and Shapes of Orbits

To visualize the actual shapes of these orbits, it is useful to translate the solution (eqn. 21) into Cartesian coordinates. With an appropriately defined Cartesian coordinate system (with the central mass at the origin), we write $R = (x^2 + y^2)^{1/2}$ and $x = R\cos(\phi - \phi_0)$ which can be used (HOMEWORK WITH HINTS IN CLASS) to translate the general solution into

$$\frac{\left[x - \frac{GM}{2\epsilon}\left(1 + \frac{2\epsilon\Lambda^2}{(GM)^2}\right)^{1/2}\right]^2}{(GM/2\epsilon)^2} - \frac{y^2}{\Lambda^2/2\epsilon} = 1. \tag{25}$$

We can now read off some important results for the three cases $\epsilon < 0$, $\epsilon > 0$ and $\epsilon = 0$.

Case A : $\epsilon < 0$: Equation (25) describes an ellipse with:

$$\begin{aligned} \text{Center at} \quad x_0 &= -\frac{GM}{2|\epsilon|}\left(1 - \frac{2|\epsilon|\Lambda^2}{(GM^2)}\right)^{1/2}, y_0 = 0 && (26) \\ \text{semi-major axis} \quad a &= \frac{GM}{2|\epsilon|} && (27) \\ \text{semi-minor axis} \quad b &= \frac{\Lambda}{\sqrt{(2|\epsilon|)}} && (28) \\ & && (29) \end{aligned}$$

Case B : $\epsilon > 0$: Equation (25) describes a hyperbola with asymptotes that have slopes

$$\frac{dy}{dx} = \pm\frac{\Lambda\sqrt{(2\epsilon)}}{GM} \tag{30}$$

Case C : $\epsilon = 0$: A little more care is needed with the analysis of this case, but we can show that the path is a parabola with closest approach on the x-axis at $R_{\min} = \lambda^2/2GM$

3.5 Kepler's Laws of Planetary Motion

Since it is such an important case (planetary orbits etc.), let's return to the $\epsilon < 0$/elliptical case. For an ellipse, the distance between a focal point and the center is called the focal length $f = \sqrt{a^2 - b^2}$ (HOMEWORK). We also define the eccentricity, $e \equiv f/a$ which characterizes the "squashed-ness" of the ellipse. We can show (ON BOARD) that

$$f = \frac{GM}{2|\epsilon|}\left[1 - \frac{\Lambda 2|\epsilon|}{(GM)^2}\right]^{1/2}, \tag{31}$$

which is exactly the same as the expression we found for the distance between the center of the ellipse and the origin (where M is located). Thus, we've shown that the gravitating mass is at one focus of the ellipse! We've just proven **Kepler's First Law**.

As an aside, we also find that

$$e = \left(1 - \frac{\Lambda^2|\epsilon|}{(GM)^2}\right)^{1/2}, \tag{32}$$

which is exactly the quantity that we've been calling "c" until now.

For **Kepler's Second Law**, we need to examine the area swept out by a line joining the orbiting particle with the gravitating mass, A. We can show that

$$\frac{dA}{dt} = \frac{1}{2}\Lambda, \tag{33}$$

and hence equal areas are swept out in equal times.

Finally, we can use this result to relate the period P of the orbit to other quantities,

$$\frac{\Lambda}{2} P = \pi a b, \tag{34}$$

and we can go onto show

$$P^2 = \frac{(2\pi)^2}{GM} a^3, \tag{35}$$

i.e., We have proven **Kepler's Third Law**.

3.6 Epicyclic Motion

It is often useful to consider the behavior of a orbit that is slightly perturbed away from some reference circular orbit (DISCUSSION IN CLASS). Consider a reference circular orbit that has radius R_0. Then, it has angular velocity $\Omega_0 = \sqrt{GM/R_0^3}$, specific angular momentum $\Lambda_0 = \sqrt{GMR_0}$, and specific energy $\epsilon = -GM/2R_0$. Thus, time dependence of the orbit is described by

$$\begin{aligned} R &= R_0 & (36) \\ \phi &= \Omega_0 t. & (37) \end{aligned}$$

Consider a perturbation,

$$\begin{aligned} R &= R_0 + x_1 & (38) \\ \phi &= \Omega_0 t + \frac{y_1}{R_0}. & (39) \end{aligned}$$

We can apply perturbation theory to derive (ON BOARD) Hill's Equation of Motion for the perturbation:

$$\ddot{x}_1 - x_1\Omega_0 - 2\Omega_0\dot{y}_1 = -2\frac{GM}{R_0^3} x_1 \tag{40}$$

$$\ddot{y}_1 + 2\dot{x}_1\Omega_0 = 0. \tag{41}$$

The solution of these coupled ordinary differential equations is

$$\begin{aligned} x_1 &= C - A\sin\Omega_0 t & (42) \\ y_1 &= Ct + 2A\cos\Omega_0 t, & (43) \end{aligned}$$

where A and C are constants.

If $C \neq 0$, the terms containing C describe the gradual azimuthal drifting away of the perturbed orbit that has a different semi-major axis (and hence period); eventually the assumptions of the perturbation analysis will fail as y_1 becomes large.

So let's set $C = 0$. The perturbed orbit executes centered elliptical motion around the guiding center of the reference orbit.

4 The Two Body Problem

SEE PROFESSOR MILLER'S NOTES.

5 The Three (and Two-Plus-One) Body Problem

5.1 General Comments

For the general system (including those with more than two bodies), the equation of motion is

$$\ddot{\mathbf{r}}_i = -G \sum_{j \neq i} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}. \tag{44}$$

We have shown that this can be solved analytically for $n = 2$ bodies. However, this does NOT have analytic solutions for for $n \geq 3$. In this case, full solutions of the equation of motion can only be obtained computationally. In general, such systems are chaotic — a tiny perturbation of the position or velocity of a particle will lead to an exponential divergence in the trajectory of the system relative to the unperturbed case.

5.2 The 2+1 Body Problem

Consider a three body system in which one of the bodies of MUCH less massive than the other two. We can then perform some instructive analysis of the system.

Let the masses of the bodies be M_1, M_2 and m. Further, suppose that $m \ll M_1, M_2$ so that we can completely neglect the gravitational force of m when considering the motions of M_1 or M_2. We say that m is a "test particle" within the gravitational fields of the two other masses. Finally, suppose that M_1 and M_2 are in circular orbits about their common center of mass, with a total separation a.

It is extremely useful to examine this system in the frame of reference rotating with the binary and with the origin placed at the center of mass — in this frame, M_1 and M_2 are at rest. The equation of motion for mass m in this rotating frame is:

$$\ddot{\mathbf{r}} = -2\Omega_0 \hat{\mathbf{z}} \times \dot{\mathbf{r}} - \Omega_0^2 \mathbf{r} - \frac{GM_1(\mathbf{r} - \mathbf{r}_1)}{|\mathbf{r} - \mathbf{r}_1|^3} - \frac{GM_2(\mathbf{r} - \mathbf{r}_2)}{|\mathbf{r} - \mathbf{r}_2|^3}, \tag{45}$$

where $\Omega_0 = [G(M_1 + M_2)/a^3]^{1/2}$ is the angular velocity of the binary. This can be translated into Cartesian coordinates (staying in the rotating frame!) where we choose to put M_1 and M_2 on the $y = 0$ line at $x = -a_1$ and $x = a_2$ respectively:

$$\ddot{x} = 2\Omega_0 \dot{y} + x\Omega_0^2 - \frac{GM_1(x + a_1)}{[(x + a_1)^2 + y^2]^{3/2}} - \frac{GM_2(x - a_2)}{[(x - a_2)^2 + y^2]^{3/2}} \tag{46}$$

$$\ddot{y} = -2\Omega_0 \dot{x} + y\Omega_0^2 - \frac{GM_1 y}{[(x + a_1)^2 + y^2]^{3/2}} - \frac{GM_2 y}{[(x - a_2)^2 + y^2]^{3/2}} \tag{47}$$

5.3 Lagrange Points

Are there places where the test mass can remain still (in the rotating frame)? Yes! To find them, put $\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0$ into equations 46. As discussed (IN CLASS), we find five such points — these are the Lagrange points. Two of them (L_4 and L_5) form equilateral triangles with M_1 and M_2, i.e., they have coordinates:

$$(x, y) = \left(a_2 - \frac{a}{2}, \pm\frac{\sqrt{3}}{2} a \right). \tag{48}$$

The other three points lie on the $y = 0$ line (i.e. the line passing through the masses M_1 and M_2). For $M_2 \ll M_1$, two of them (L_1 and L_2) lie close to the smaller mass,

$$(x, y) \approx \left(a_2 \pm a \left[\frac{M_2}{3M_1} \right]^{1/3}, 0 \right), \tag{49}$$

and the last Lagrange point lies at $(x, y) \approx (-a, 0)$, i.e., this point lies on the "opposite side of the orbit" of M_2 around M_1.

All of the in-line Lagrange points are unstable... the test particle will accelerate away from this location if given a slight push. L_4 and L_5 are stable provided $M_2 < M_1/26$.

5.4 Effective Potential of the 2+1 Body Problem

We can rewrite the equation of motion as

$$\ddot{\mathbf{r}} = -2\Omega_0 \hat{\mathbf{z}} \times \dot{\mathbf{r}} - \nabla \Phi_{\rm eff}, \tag{50}$$

where

$$\Phi_{\rm eff} = -\frac{\Omega_0^2}{2} r^2 - \frac{GM_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{GM_1}{|\mathbf{r} - \mathbf{r}_2|}, \tag{51}$$

is the *effective potential* of the problem. This gives the acceleration of a stationary test particle, including the effects of the centrifugal force.

5.5 Astrophysical Significant of Lagrange Points

1. TROJAN ASTEROIDS — a group of asteroids "trapped" in the L4/L5 Lagrange points of the Sun-Jupiter system.

2. MASS TRANSFER IN BINARY STARS — if one star fills the critical iso-potential surface, mass is transferred across the L1 Lagrange Point towards the other star,

3. SPACE-SCIENCE — L1/L2 are attractive places to put scientific spacecraft.

6 The N-body Problem

We now look at the general N-body problem. Of course, as with 3-bodies, a full solution of this problem is not possible with analytic techniques — one must resort to numerical solutions of the equations if a full solution is required. However, we can still deduce some interesting and important aspects of N-body systems even without a computer.

6.1 The Virial Theorem

We consider a system with N bodies interacting purely gravitationally — all non-gravitational forces are neglected and hence we do not account for the possibility of collisions between objects. The full equation of motion for the N-body system are:

$$m_i\ddot{\mathbf{r}}_i = -\sum_{\mathrm{J}\neq i} \frac{Gm_i m_j(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3}. \quad (52)$$

Now, we define the “scalar moment of inertia” as

$$I \equiv \sum_i m_i \mathbf{r}_i \cdot \mathbf{r}_j. \quad (53)$$

Suppose that the system is in a state of statistical equilibrium so that $\langle \ddot{I} \rangle = 0$ (where the $\langle . \rangle$ denotes a time average), then

$$\langle W \rangle + 2\langle K \rangle = 0, \quad (54)$$

where W and K are the gravitational potential and kinetic energy respectively given by

$$W = -\sum_i \sum_{j=1}^{i-1} \frac{Gm_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}, \quad (55)$$

$$K = \sum_i \frac{1}{2} m_i |\dot{\mathbf{r}}|^2. \quad (56)$$

The result given in equation-54 is known as the **Virial Theorem** and is an important theoretical tool for understanding N-body systems.

6.2 Masses of galaxies from the Virial Theorem

Consider a galaxy as an N-body system. For illustration purposes, let us suppose that all of the stars have the same mass m and an isotropic velocity distribution (these assumptions can be generalized straightforwardly). The isotropy assumption, in particular, means that we are really talking about an elliptical galaxy. Then, from the Virial Theorem, we can show that

$$M = \frac{6\bar{R}\sigma^2}{G}, \quad (57)$$

where σ^2 is the one-dimensional velocity dispersion of the stars and $\bar{R}$ is defined by

$$\frac{1}{\bar{R}} \equiv \frac{2}{N(N-1)} \sum_{\mathrm{pairs}} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \quad (58)$$

This is a powerful result — it allows us to determine the mass of the galaxy by just measuring its size and velocity dispersion.

Let's get a little more specific. Assume that the galaxy has a density distribution $\rho(r) \propto r^{-\alpha}$ out to a radius $\mathcal{R}$ (and zero for $R > \mathcal{R}$). Then, with some work to determine $\bar{R}$, we can write this relation as

$$M = \frac{5\sigma^2\mathcal{R}}{G}\frac{1-2\alpha/3}{1-\alpha/3} \tag{59}$$

6.3 Collapse of cosmic structure from the Virial Theorem

The early Universe was rather homogeneous. Large-scale structure in the Universe (from dwarf galaxies up to giant clusters of galaxies) are believed to have formed by the gravitational collapse of regions of the Universe that were slightly overdense relative to the average density.

By combining the Virial Theorem with the conservation of energy, we can analyze the collapse of overdense (dark) matter regions. If we imagine some overdense region with initial radius R_i, the Virial Theorem implies that it will collapse to form a "virtualized" structure that has a final radius $R_f \sim R_i/2$. Hence the density of collapsed structures will be about 8 times the average density of the Universe at the time of collapse.

6.4 Two-body relaxation

The process by which collision-less gravitational systems collapse and come into statistical equilibrium is interesting in its own right and is known as **violent relaxation** (WE WILL DISCUSS THIS IN CLASS!). However, the statistical equilibrium that results from violent relaxation is not unique — there can be gradual evolution to a higher entropy state. An important process driving this evolution are the gravitational interactions between individual particles — this kind of evolution is called two-body relation.

Consider the interaction of two particles in the system. Suppose that, relative to one particle, the other has a velocity v and an impact parameter b. We can show (IN CLASS) that the fractional change in the kinetic energy of the second particle is

$$\frac{\Delta E}{E} \approx \left(\frac{2GM}{v^2 b}\right)^2. \tag{60}$$

If the number density of the system is n, the total rate with which a particle is interacting with other particles in a range of impact parameters $b \to b + db$ is

$$\frac{d\mathcal{N}}{dt} = 2\pi b n v \, db. \tag{61}$$

By integrating the rate of energy loss over the impact parameters, we can calculate the relaxation timescale (i.e. the time over which the energy of a typical particle changes by an amount comparable to its initial energy):

$$t_{\rm rel} \sim \frac{v^3}{(GM)^2 n \ln(b_{\rm max}/b_{\rm min})}. \tag{62}$$

Note that the dependence on b_{max} and b_{min} is very weak. We say that $b_{\mathrm{min}} \sim n^{-1/3}$ and that $b_{\mathrm{max}} \sim R$. Putting everything together and doing some algebra, we get the commonly quoted result for the relaxation time,

$$t_{\mathrm{rel}} \approx \frac{0.1N}{\ln N} t_{\mathrm{cross}}, \tag{63}$$

where we have defined the crossing time of the system as $t_{\mathrm{cross}} \equiv R/v$.

In class, we show that globular clusters have had plenty of time to relax, (whole) galaxies have not relaxed, and galaxy clusters are marginally relaxed.

6.5 Dynamical Friction

Due to two-body relaxation, a massive body moving through a sea of smaller bodies will tend to give up energy to those smaller bodies. This effect is called *dynamical friction*. The massive body M feels a force opposing its motion V through (relative to) the sea of small bodies with density ρ. An analysis similar to that used for the relaxation time gives

$$F_{\mathrm{DF}} = -4\pi\rho \frac{(GM)^2}{V^2} \ln\left(\frac{b_{\mathrm{max}}}{b_{\mathrm{min}}}\right) \tag{64}$$

It is interesting to note that F_{DM} is proportional to M^2 and V^{-2} (hence more massive and also slower bodies are much more strongly affected).

Dynamical friction is important for a variety of astrophysical phenomena (DISCUSSION IN CLASS).