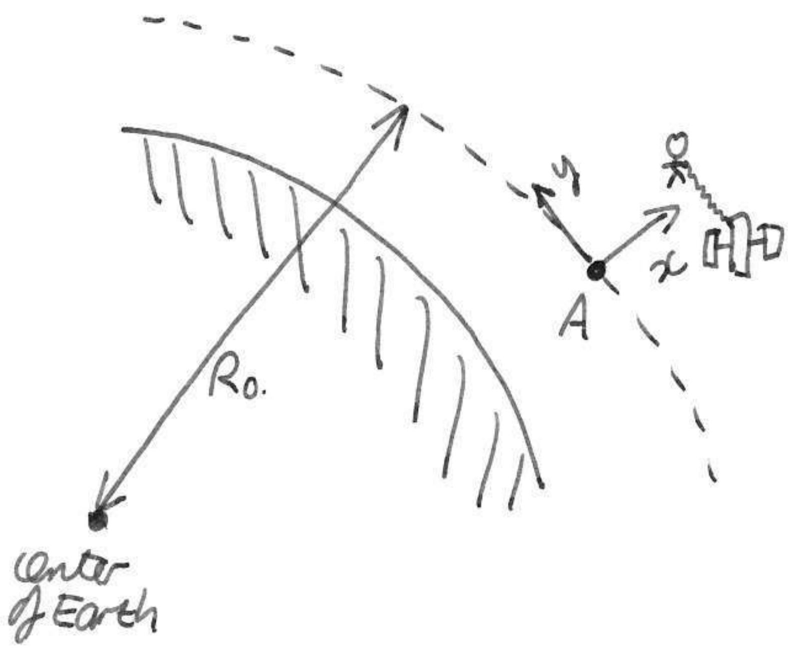
(3) Astronaut on a bungee cord



A is "guiding center" of the perturbed orbits.

(a) Let $\underline{r}_1$ be pos^n vector of astronaut
$\underline{r}_2$ " " " " spacestation

Then the full equation of motion for the astronaut is

$$m_1 \ddot{\underline{r}}_1 = -\frac{GMm_1}{r_1^2}\hat{\underline{r}}_1 + k(\underline{r}_2 - \underline{r}_1)$$

Following approach used in class, we split into radial and tangential components:

$$\ddot{R}_1 - R_1\dot{\varphi}_1 = -\frac{GM}{R_1^2} + \frac{k}{m_1}(\underline{r}_2 - \underline{r}_1)\cdot\hat{\underline{R}} \quad \cdots (1)$$

$$R_1\ddot{\varphi}_1 + 2\dot{R}_1\dot{\varphi}_1 = \frac{k}{m_1}(\underline{r}_2 - \underline{r}_1)\cdot\hat{\underline{\varphi}} \quad \cdots (2)$$

As in class, we set

$$R_1 = R_0 + x_1$$

$$\varphi_1 = \Omega_0 t + y_1/R_0$$

where $\Omega_0 = (GM/R_0^3)^{1/2}$ is ang. velocity of reference circular orbit

This means that

$$\underline{r}_1 = (R_0 + x_1)\hat{\underline{R}} + y_1 \hat{\underline{\varphi}}$$
$$\underline{r}_2 = (R_0 + x_2)\hat{\underline{R}} + y_2 \hat{\underline{\varphi}}$$

So equation 1 becomes

$$\ddot{x}_1 - (R_0 + x_1)(\Omega_0 + \dot{y}_1/R_0)^2 = -\frac{GM}{(R_0+x_1)^2} + \frac{k}{m_1}(x_2 - x_1)$$

$$\Rightarrow \ddot{x}_1 - R_0\Omega_0^2 - \Omega_0^2 x_1 - 2R_0 \dot{y}_1 \Omega_0 / R_0$$
$$= -\frac{GM}{R_0^2} + 2x_1 \frac{GM}{R_0^3} + \frac{k}{m_1}(x_2 - x_1)$$

(dropping terms like $\dot{y}_1^2$ and $x_1\dot{y}_1$ since they are small, and Taylor expanding $GM/(R_0+x_1)^2$ to first order)

$$\Rightarrow \ddot{x}_1 - \cancel{R_0\Omega_0^2} - \Omega_0^2 x_1 - 2\Omega_0 \dot{y}_1 = -\cancel{R_0\Omega_0^2} + 2x_1\Omega_0^2 + \frac{k}{m_1}(x_2 - x_1)$$

where we have used fact that $\Omega_0^2 = GM/R_0^3$

$$\boxed{\Rightarrow \ddot{x}_1 - 3\Omega_0^2 x_1 - 2\Omega_0 \dot{y}_1 - \frac{k}{m_1}(x_2 - x_1) = 0} \quad \cdots 3$$

Equation 2 becomes

$$(R_0 + x_1)\ddot{y}_1/R_0 + 2\dot{x}_1(\Omega_0 + \dot{y}_1/R_0) = \frac{k}{m_1}(y_2 - y_1)$$

$$\boxed{\Rightarrow \ddot{y}_1 + 2\dot{x}_1\Omega_0 - k/m_1 (y_2 - y_1) = 0} \quad \cdots 4$$

where we have neglected terms like $x_1\ddot{y}_1$ & $\dot{x}_1\dot{y}_1$ since they are small in comparison.

Equations 3 & 4 are the required equations of motion

(b) The derivation of the equation of motion for the spacestation will look exactly the same except that the labels "1" and "2" will be transposed. So,

$$\boxed{\begin{aligned} &\ddot{x}_2 - 3\Omega_0^2 x_2 - 2\Omega_0 \dot{y}_2 - \frac{k}{m_2}(x_1 - x_2) = 0 \quad \cdots (5) \\ &\ddot{y}_2 + 2\dot{x}_2\Omega_0 - \frac{k}{m_2}(y_1 - y_2) = 0 \quad \cdots (6) \end{aligned}}$$

Subtract (5) from (3) gives

$$\ddot{x}_1 - \ddot{x}_2 - 3\Omega_0^2 x_1 + 3\Omega_0^2 x_2 - 2\Omega_0 \dot{y}_1 + 2\Omega_0 \dot{y}_2$$
$$- \frac{k}{m_1}(x_2 - x_1) + \frac{k}{m_2}(x_1 - x_2) = 0$$

$$\Rightarrow (\ddot{x}_1 - \ddot{x}_2) - 3\Omega_0^2(x_1 - x_2) - 2\Omega_0(\dot{y}_1 - \dot{y}_2)$$
$$+ k(x_1 - x_2)\left(\frac{1}{m_2} + \frac{1}{m_1}\right) = 0$$

Let $\Delta x = x_1 - x_2$ and $\Delta y = y_1 - y_2$. So, this last eq^n reads

$$\Delta\ddot{x} - 3\Omega_0^2 \Delta x - 2\Omega_0 \Delta\dot{y} + k\Delta x\left(\frac{1}{m_2} + \frac{1}{m_1}\right) = 0$$

But $m_2 >> m_1$ so $\frac{1}{m_2} << \frac{1}{m_1}$. Thus, neglecting $\frac{1}{m_2}$ term gives

$$\boxed{\Delta\ddot{x} - \left(3\Omega_0 - \frac{k}{m_1}\right)\Delta x - 2\Omega_0 \Delta\dot{y} = 0} \quad \cdots (7)$$

Subtract (6) from (4)

$$\ddot{y}_1 - \ddot{y}_2 + 2\dot{x}_1\Omega_0 - 2\dot{x}_2\Omega_0 - \frac{k}{m_1}(y_2 - y_1) + \frac{k}{m_2}(y_1 - y_2) = 0$$

$$\Rightarrow \Delta\ddot{y} + 2\Omega_0 \Delta\dot{x} + k\Delta y\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = 0$$

$$\Rightarrow \boxed{\Delta\ddot{y} + 2\Omega_0 \Delta\dot{x} + \frac{k}{m_1}\Delta y = 0} \quad \cdots (8)$$

AS REQUIRED

(c) k/m_1 has units of frequency-squared, so it should be compared with the only other frequency in the problem, Ω_0. Suppose that

$$\frac{k}{m_1} \gg \Omega_0^2$$

Then, we can essentially neglect the terms containing Ω_0 in the equation of motion. So, we have

$$\Delta\ddot{x} + \frac{k}{m_1}\Delta x = 0$$

$$\Delta\ddot{y} + \frac{k}{m_1}\Delta y = 0$$

This is the simple harmonic oscillator eq^n... both Δx and Δy execute SHO with frequency $\omega = \sqrt{k/m_1}$, ie

$$\Delta x = A\cos(\omega t + \phi_A)$$
$$\Delta y = B\cos(\omega t + \phi_B)$$
$$\omega = \sqrt{k/m_1} \text{ and } A, B, \phi_A, \phi_B \text{ constants}$$

So, the astronaut executes "orbits" around the space-station. The space-station is at the center of the elliptical orbit and the period of the orbit is $\omega = \sqrt{k/m_1}$

(d) Need to combine eq^{n}s to eliminate Δy. Start with

$$\Delta\ddot{x} - (3\Omega_0^2 - k/m_1)\Delta x - 2\Omega_0\Delta\dot{y} = 0 \ \cdots$$

$$\Rightarrow \Delta\dddot{x} - (3\Omega_0^2 - k/m_1)\Delta\dot{x} - 2\Omega_0\Delta\ddot{y} = 0 \quad \text{differentiating wrt time.}$$

But, we know that

$$\Delta\ddot{y} = -2\Omega_0\Delta\dot{x} - \frac{k}{m_1}\Delta y$$

Subst. this into eqⁿ for $\Delta\dddot{x}$ gives

$$\Delta\dddot{x} - (3\Omega_0^2 - k/m_1)\Delta\dot{x} - 2\Omega_0\left(-2\Omega_0\Delta\dot{x} - \frac{k}{m_1}\Delta y\right) = 0$$

$$\Rightarrow \Delta\dddot{x} + (\Omega_0^2 + k/m_1)\Delta\dot{x} + 2\Omega_0\frac{k}{m_1}\Delta y = 0$$

$$\Rightarrow \Delta\ddddot{x} + (\Omega_0^2 + k/m_1)\Delta\ddot{x} + 2\Omega_0\frac{k}{m_1}\Delta\dot{y} = 0$$

But, from (7) we have

$$2\Omega_0\Delta\dot{y} = \Delta\ddot{x} - (3\Omega_0^2 - k/m_1)\Delta x$$

Subst. this into eqⁿ for $\Delta\ddddot{x}$ gives

$$\Delta\ddddot{x} + (\Omega_0^2 + k/m_1)\Delta\ddot{x} + \frac{k}{m_1}\left(\Delta\ddot{x} - \left[3\Omega_0^2 - \frac{k}{m_1}\right]\Delta x\right) = 0$$

$$\boxed{\Rightarrow \Delta\ddddot{x} + \left(\Omega_0^2 + \frac{2k}{m_1}\right)\Delta\ddot{x} - \frac{k}{m_1}\left(3\Omega_0^2 - \frac{k}{m_1}\right)\Delta x = 0}$$

This is the required differential equation

To get characteristic equation, substitute in $\Delta x = A e^{\alpha t}$.

Then
$$\Delta\dot{x} = A\alpha e^{\alpha t} = \alpha\Delta x$$
$$\Delta\ddot{x} = \alpha\Delta\dot{x} = \alpha^2\Delta x$$
$$\text{etc}$$
$$\vdots$$
$$\Delta\ddddot{x} = \alpha^4\Delta x$$

So, we get

$$\alpha^4\Delta x + \left(\Omega_0^2 + \frac{2k}{m_1}\right)\alpha^2\Delta x - \frac{k}{m_1}\left(3\Omega_0^2 - \frac{k}{m_1}\right)\Delta x = 0$$

$$\boxed{\Rightarrow \alpha^4 + \left(\Omega_0^2 + \frac{2k}{m_1}\right)\alpha^2 - \frac{k}{m_1}\left(3\Omega_0^2 - \frac{k}{m_1}\right) = 0}$$

AS REQUIRED.

(e) Suppose that the characteristic equation has a positive real root α_0. Then the general solution of the differential equation will have a term that looks like

$$A e^{\alpha_0 t}$$

But $e^{\alpha_0 t} \Rightarrow \infty$ as $t \Rightarrow \infty$ and so solution blows up!!

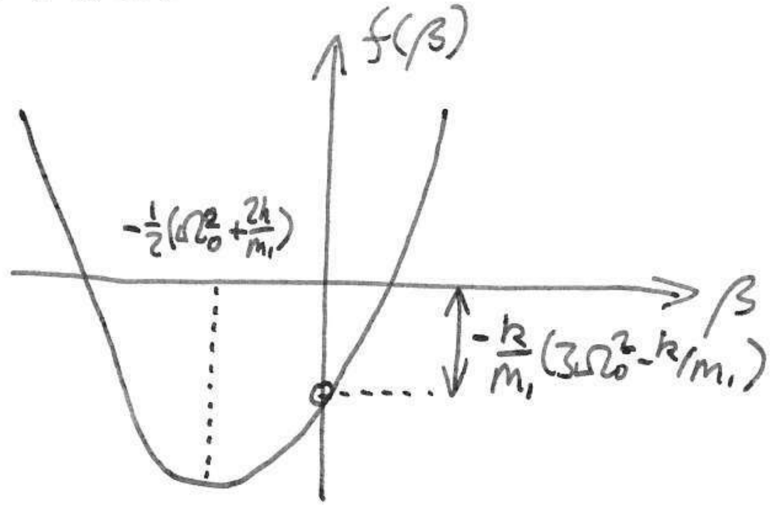
View characteristic equation as a quadratic in $\beta \equiv \alpha^2$. Then there will be a positive real root if and only if the quadratic for β has a positive real root.

(f) Look at characteristic equation as quadratic in $\beta \equiv \alpha^2$.

$$\beta^2 + \left(\Omega_0^2 + \frac{2k}{m_1}\right)\beta - \frac{k}{m_1}\left(3\Omega_0^2 - \frac{k}{m_1}\right) = 0$$

ie $f(\beta) = 0$, $f(\beta) = \beta^2 + \left(\Omega_0^2 + \frac{2k}{m_1}\right)\beta - \frac{k}{m_1}\left(3\Omega_0^2 - \frac{k}{m_1}\right)$

Lets graph this :



The equation ~~max~~ $f(\beta) = 0$ has real roots if and only if

$$\frac{k}{m_1}\left(3\Omega_0^2 - k/m_1\right) \geq 0$$

$$\Rightarrow k \leq 3\Omega_0^2 m_1$$

Thus, if the bungee cord is weak ($k < 3\Omega_0^2 m$), then the general equation of the differential equation will have an exponentially diverging piece.

The astronaut will drift away from the station at an exponential rate!!