Part C : Quantum Physics

1 Particle-wave duality

1.1 The Bohr model for the atom

We begin our discussion of quantum physics by discussing an early idea for atomic structure, the Bohr model. While this relies on a rather arbitrary assumptions (as we shall see), it does demonstrate the power of the idea of quantization and was the first model of the atom that was truly predictive.

In the model model, the atom consists of a massive nucleus with charge $+Ze$ surrounded by much lighter electrons (charge $-e$) that are in circular orbits. We proceed using classical Newtonian arguments. The equation of motion for the orbiting electron is

$$\frac{m_e v^2}{r} = \frac{Ze^2}{4\pi\epsilon_0 r^2}, \tag{1}$$

which tells us that the angular momentum of the electron is

$$L \equiv m_e v r = \sqrt{\frac{Ze^2 m_e r}{4\pi\epsilon_0}}. \tag{2}$$

The energy of the electron is

$$E = \frac{1}{2} m_e v^2 - \frac{Ze^2}{4\pi\epsilon_0 r}, \tag{3}$$

which, using the equation of motion, can be written in terms of just r:

$$E = -\frac{Ze^2}{8\pi\epsilon_0 r} \tag{4}$$

Bohr stated that, whenever electrons move one orbit to another, the change in energy is released (or absorbed) as a packet of e/m radiation with frequency proportional to the change in energy (in modern language, we would say that a photon is emitted or absorbed). But we know that atoms have well-defined emission and absorption lines, so it seems as if only very particular energy jumps are allowed. Why?

Bohr asserted that the angular momentum of the electron is quantized,

$$L = n\hbar, \tag{5}$$

where $\hbar$ is the reduced Planck constant ($h/2\pi$) and $n = 1, 2, 3...$ Let us follow this suggestion through to its conclusion. Inserting equation 2 into equation 5, we deduce that only particular values of r are allowed,

$$r_n = \frac{4\pi\epsilon_0\hbar^2}{Ze^2 m_e} n^2. \tag{6}$$

This in turn means that only particular energy levels are allowed,

$$E_n = -\frac{Z^2 e^4 m_e}{32\pi^2\epsilon_0^2\hbar^2}\frac{1}{n^2} \tag{7}$$

Suppose that an electron jumps from a level with $n = n_1$ to $n = n_2$. The change in energy is

$$\Delta E_{n_1\to n_2} = E_{n_2} - E_{n_1} = \frac{Z^2 e^4 m_e}{32\pi^2\epsilon_0^2\hbar^2}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) \tag{8}$$

If $n_2 < n_1$, then the electron loses energy (it falls deeper into the electrostatic potential) and a photon is emitted (emission line). If $n_1 < n_2$, the electron gains energy and it must have absorbed a photon (absorption line). The wavelengths of the corresponding emission/absorption lines are at

$$\lambda = \frac{hc}{|\Delta E_{n_1\to n_2}|}. \tag{9}$$

1.2 Matter waves and particle-wave duality

In 1923, de Broglie built upon Einstein's ideas of photons to make a radical proposal — that matter particles have wave-like properties, and that the corresponding wavelength is related to the particles momentum by $\lambda = h/p$. Vectorially, we have $\mathbf{p} = \hbar\mathbf{k}$ where $\mathbf{k}$ is the wavenumber vector. This idea was verified by the detection of electron diffraction in 1927.

We can now understand the Bohr angular momentum quantization in terms of de Broglie waves. Consider an electron in an atom, orbiting the nucleus at radius r. In order to "fit", the wavelength of the electron must be an integer number of multiples of the circumference of the orbit, i.e.

$$2\pi r = n\lambda. \tag{10}$$

But, since $\lambda = h/p$, this formula can be easily re-arranged to give

$$rp = h\hbar, \tag{11}$$

and we recognize the left-hand side of this as just the angular momentum L.

2 Degeneracy pressure and compact stars

Quantum physics is crucial for an understanding of white dwarfs and neutron stars. The material in these stars is rather hot and so all atoms are fully ionized. Thus we must examine the dynamics of "free particles". This brings us to the classic discussion of...

2.1 A particle in a box

Consider a particles in a cubic box of side length L. Suppose that the walls of the box are impenetrable. The particle is described by a 3-d wave that needs to fit inside the box. Thus, we need:

$$\begin{aligned} n_x \frac{\lambda_x}{2} &= L, & (12)\\ n_y \frac{\lambda_y}{2} &= L, & (13)\\ n_z \frac{\lambda_z}{2} &= L, & (14)\\ & & (15) \end{aligned}$$

where n_x, n_y, n_z are positive integers. Using de Broglie's formula, this gives a quantization of the particle's momentum components

$$\begin{aligned} p_x &= n_x \frac{h}{2L}, & (16)\\ p_y &= n_y \frac{h}{2L}, & (17)\\ p_z &= n_z \frac{h}{2L}, & (18) \end{aligned}$$

and hence a quantization of the energy,

$$E = \frac{p^2}{2m} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n + z^2) \qquad (\mathrm{non-relativistic}) \qquad (19)$$

$$E = cp = \frac{ch}{2L}(n_x^2 + n_y^2 + n + z^2)^{1/2} \qquad (\mathrm{ultra-relativistic}). \qquad (20)$$

Some notes:

1. For any finite L, the allowed energy levels are quantized.
2. As $L \to \infty$, the energy levels get closer together and (sort of) approach a continuum.
3. There is a minimum, non-zero, allowed energy!
4. The "state" of the particle can be labeled by the three "quantum numbers"; n_x, n_y, n_z.

2.2 Fermions, Bosons and Pauli exclusion

There are two basic types of particles in nature called Bosons and Fermions.

Bosons (e.g. photons, gravitons, gluons) have spin angular momenta that are integer multiples of $\hbar$. For example, a photon has a spin of $\hbar$; a graviton has a spin of $2\hbar$. There is no restriction on the number of bosons that can be in a given quantum state... indeed, bosons like being in the same quantum state!

Fermions (e.g., electrons, quarks, protons, neutrons) have spins that are half-integer multiples of $\hbar$. For examples, our familiar particles (electrons, protons, neutrons) all have spin $\frac{1}{2}\hbar$. A given measurement of the spin will find it in one of two states, "up" ($S = \frac{1}{2}\hbar$) or "down" ($S = -\frac{1}{2}\hbar$). No more than one fermion can occupy a given quantum state (including the spin as one of the quantum numbers) — this is the Pauli exclusion principle.

2.3 Degenerate matter and the Fermi-energy

Suppose that we have N spin-1/2 fermions (e.g. electrons) in a box and that they are free and non-interacting. They must, however, obey the Pauli exclusion principle. So, we can have at most 2 electrons in the $(n_x, n_y, n_z) = (1, 1, 1)$ state, 2 electrons in the $(n_x, n_y, n_z) = (1, 1, 2)$ state, 2 electrons in the $(n_x, n_y, n_z) = (1, 2, 1)$ state etc. etc.

If the system is very cold, the electrons will all attempt to occupy the lowest possible energy levels but, of course, they are now allowed to all occupy the lowest energy level. They will "stack up" in order of increasing E, which means in order of increasing $(n_x^2+n_y^2+n_z^2)^{1/2}$. In other words, they will stack up into a spherical ball in $\mathbf{n}$-space and hence $\mathbf{p} - space$. The maximum momentum achieved (i.e. the momentum of this outer "shell" in momentum space) is

$$p_F = \hbar(3\pi^2 n_e)^{1/3}, \tag{21}$$

where n_e is the number density of the electrons. This is known as the Fermi-momentum. The corresponding energy,

$$E_F = \frac{p_F^2}{2m_e} = \frac{\hbar^2}{2m_e}(3\pi^2 n_e)^{2/3} \qquad \text{(non-relativistic particles)} \tag{22}$$

$$E_F = p_F c = \hbar c(3\pi^2 n_e)^{1/3} \qquad \text{(relativistic particles)}, \tag{23}$$

is known as the Fermi-energy.

2.4 Degeneracy Pressure

The Pauli exclusion principle means that the distribution function of cold degenerate matter is particularly simple,

$$f_{\mathbf{p}}(\mathbf{p}) \;=\; A, \text{constant} \quad \text{for } |\mathbf{p}| < p_F \tag{24}$$

$$f_{\mathbf{p}}(\mathbf{p}) \;=\; 0 \quad \text{otherwise.} \tag{25}$$

where $f_{\mathbf{p}}(\mathbf{p})\, dp_x\, dp_y\, dp_z$ is the probability of finding an electron in state with momentum in range $p_x \to p_x + dp_x, p_y \to p_y + dp_y, p_z \to p_z + dp_z$. We determine A by the normalization condition:

$$\int f_{\mathbf{p}}(\mathbf{p})\, d^3\mathbf{p} = 1 \Rightarrow A = \frac{2}{\pi^3\hbar^3 n_e} \tag{26}$$

. From the distribution function, we can now calculate the pressure using our machinery of statistical mechanics. Assuming a non-relativistic relation between velocity and momentum ($v_x = p_x/m_e$), this goes as:

$$\mathcal{P} \;=\; \int\int\int n_e f(\mathrm{v}) m_e v_x^2\, dv_x\, dv_y\, dv_z \tag{27}$$

$$= \frac{n_e}{m} \int \int \int p_x^2 f_{\mathbf{p}}(\mathbf{p}) \, dp_x \, dp_y \, dp_z \tag{28}$$

$$= \frac{(3\pi^2)^{2/3}\hbar^2}{5m_e} n_e^{5/3} \tag{29}$$