Two-body problem

Our treatment of one-body gravity is reasonable when one object dominates the gravity in a system, as the Sun does in the Solar System. This, however, is not always the case. **Ask class:** can they think of an example in which the mass ratio is not large? Binary stars are a perfect example. In fact, most stars are in binaries (i.e., if for three primaries there are two secondaries, on average, then four out of the five stars are in binaries). We therefore need to know what happens when two comparable-mass objects orbit each other. We will assume that both masses act like points, to avoid complications initially.

As a first step, let's define a special coordinate system: the *center of momentum* coordinate system. We do this because the only net change to the momentum of a system occurs in response to external forces. Therefore, by moving into the center of momentum frame, we separate out the uninteresting overall motion of the system. Let m_1 be at $\mathbf{r}_1$ and m_2 be at $\mathbf{r}_2$, where we make no restrictions on the relative magnitudes of m_1 and m_2. These radius vectors are defined in some inertial observer's frame. Let's define

$$\mathbf{r} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \ . \tag{1}$$

This is the coordinate of the center of mass of the system, in the chosen observer's frame. The total momentum of the system is $\mathbf{p}_{\rm tot} = m_1\dot{\mathbf{r}}_1 + m_2\dot{\mathbf{r}}_2 = (m_1+m_2)\dot{\mathbf{r}} \equiv M_{\rm tot}\dot{\mathbf{r}}$. Therefore, the center of mass is also the center of momentum.

Let us now define the position with respect to the center of mass as $\mathbf{R}_1 \equiv \mathbf{r}_1 - \mathbf{r}$ and $\mathbf{R}_2 \equiv \mathbf{r}_2 - \mathbf{r}$, and the relative positions of the two masses as $\mathbf{R} \equiv \mathbf{R}_1 - \mathbf{R}_2 = \mathbf{r}_1 - \mathbf{r}_2$. The $\mathbf{F} = m\mathbf{a}$ equations of motion are then

$$\begin{aligned} m_1\ddot{\mathbf{r}}_1 &= -\tfrac{Gm_1m_2}{R^3}\mathbf{R} \\ m_2\ddot{\mathbf{r}}_2 &= \tfrac{Gm_1m_2}{R^3}\mathbf{R} \ . \end{aligned} \tag{2}$$

Adding these together we get

$$\begin{aligned} m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 &= 0 \\ (m_1+m_2)\ddot{\mathbf{r}} &= 0 \\ (m_1+m_2)\dot{\mathbf{r}} &= \text{constant} \ . \end{aligned} \tag{3}$$

Ask class: what does this mean? It means that the total momentum $\mathbf{p}_{\rm tot} = (m_1+m_2)\dot{\mathbf{r}}$ is constant, which it had better be! It is useful to do these types of checks on occasion during a derivation.

Now let's multiply the first of our equations of motion by m_2, the second by m_1, and subtract:

$$m_1m_2(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = -\frac{Gm_1m_2(m_1+m_2)}{R^3}\mathbf{R} \ . \tag{4}$$

Now we recognize that since $\mathbf{R} = \mathbf{R}_1 - \mathbf{R}_2 = \mathbf{r}_1 - \mathbf{r}_2$, the second time derivative is just $\ddot{\mathbf{R}}$, so we have finally

$$\ddot{\mathbf{R}} = -\frac{G(m_1 + m_2)}{R^3}\mathbf{R}\ . \tag{5}$$

Ask class: what does this equation mean? It means that the two-body problem reduces *exactly* to the one-body problem, except that the mass is now the total mass and the vector **R** that is changing doesn't represent the actual position of a body, but rather the separation vector of the two bodies. Wow! This is pretty cool, since it means that we can now transfer all the insight we gained in one-body orbits to two-body orbits.

Note that the relative motion of the objects is *independent* of the initial coordinate system we used (the one in which the positions of the bodies were $\mathbf{r}_1$ and $\mathbf{r}_2$). This has to be the case; it's an example of a symmetry. If it were otherwise, then, for example, the orbits of planets in the Solar System would depend on which alien happened to be observing us at a given time!

What if we want the motion of each individual body? First, we solve the equivalent one-body problem for **R**. We then use

$$\begin{aligned} m_1\mathbf{r}_1 + m_2\mathbf{r}_2 &= (m_1 + m_2)\mathbf{r} \\ \mathbf{r}_1 - \mathbf{r}_2 &= \mathbf{R}\ . \end{aligned} \tag{6}$$

Solving for $\mathbf{r}_1$ and $\mathbf{r}_2$ we get

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r} + m_2/(m_1 + m_2)\mathbf{R} = \mathbf{r} + \tfrac{\mu}{m_1}\mathbf{R} \\ \mathbf{r}_2 &= \mathbf{r} - m_1/(m_1 + m_2)\mathbf{R} = \mathbf{r} - \tfrac{\mu}{m_2}\mathbf{R} \end{aligned} \tag{7}$$

where as in the last class we have defined the *reduced mass* $\mu \equiv m_1 m_2/(m_1 + m_2)$. If you look at the motions of the two bodies in detail, you find that each of them moves in an ellipse with one focus being at the center of mass of the system.

The total energy of the system, as seen in the frame of our original inertial observer, is

$$E = \frac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \frac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - \frac{Gm_1m_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\ . \tag{8}$$

After some algebraic manipulation, you can put this in the form

$$E = \frac{1}{2}M_{\rm tot}|\dot{\mathbf{r}}|^2 + \frac{1}{2}\mu|\dot{\mathbf{R}}|^2 - \frac{G\mu M_{\rm tot}}{|\mathbf{R}|}\ , \tag{9}$$

where $M_{\rm tot} \equiv m_1 + m_2$. **Ask class:** what is the physical meaning of the first term? It's the kinetic energy of the total mass $M_{\rm tot}$, moving as if it were a particle at the center of mass of the system. **Ask class:** what is the physical meaning of the last two terms? It's the energy of an equivalent one-body system, as if there were a single particle of mass μ.

Likewise, the total angular momentum of the system, as seen in the original inertial frame, is

$$\mathbf{L} = m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2 \ . \tag{10}$$

After some fiddling, you can get this into the form

$$\mathbf{L} = M_{\rm tot} \mathbf{r} \times \dot{\mathbf{r}} + \mu \mathbf{R} \times \dot{\mathbf{R}} \ . \tag{11}$$

Ask class: what are the physical meanings of these two terms? The first one is the angular momentum of the center of mass, and the second is the angular momentum of the "reduced mass" equivalent one-body system.

Traditionally, one can detect and observe binaries in three ways. Some are eclipsing, meaning that the photometric brightness of the system varies periodically. Some are visual, meaning that one can see the two stars orbit around each other. Some are spectroscopic, meaning that one can see the positions of the spectral lines vary as the stars move. With perfect instruments, one could in principle see all three aspects of binary motion, but I don't know of an example.

Let us apply our binary dynamics to a practical situation. Suppose you are looking at a spectroscopic binary with high resolution, but that you can only get a good spectrum for one of the two objects (call it object 1, of mass m_1). **Ask class:** what quantities can you measure that bear on the masses of one star or the other? From the variation of the spectrum, you can get the orbital period and can also get the component of the velocity of the star you're measuring that is in the direction of your line of sight. That's because transverse Doppler shifts are tiny compared to those towards you or away from you. Let's say that the orbital axis of the system makes an angle i with respect to your line of sight. Then the maximum radial velocity you measure turns out to be

$$Q = \frac{\mu}{m_1} \frac{a}{P} \sin i \ . \tag{12}$$

Ask class: does this have the right units? We want a velocity, and a/P is a distance over a time with the other factors dimensionless, so that works. **Ask class:** does this have the right limits? If $i = 0$ there is no radial velocity, so Q should be 0 and it is. If $m_2 \ll m_1$ then object 1 is barely moved by object 2, so Q should be small. Indeed, since $\mu \equiv m_1 m_2/(m_1 + m_2)$, if $m_2 \ll m_1$ then $\mu \approx m_2$ and $\mu/m_1 \ll 1$, so Q is small. Now, the orbital period P of the system is given by

$$P^2 = \frac{(2\pi)^2 a^3}{G(m_1 + m_2)} \tag{13}$$

so one can write

$$Q^3 = \left(\frac{\mu}{m_1}\right)^3 \frac{a^3}{P^3} \sin^3 i = \left(\frac{\mu}{m_1}\right)^3 \frac{P^2 G(m_1 + m_2)}{(2\pi)^2 P^3} \sin^3 i \ . \tag{14}$$

Since P and Q are observable, then from the orbit one can deduce the quantity

$$\left(\frac{\mu}{m_1}\right)^3 (m_1 + m_2) \sin^3 i = \frac{m_2^3}{(m_1 + m_2)^2} \sin^3 i \ . \tag{15}$$

This is called the *mass function.* For a given mass function, the mass m_2 is minimized by setting $\sin i = 1$ and $m_1 = 0$, in which case the mass function is just m_2. Therefore, the mass function is the minimum possible mass of m_2. If $m_1 \gg m_2$ is known, the mass function gives a value for $m_2 \sin i$. If m_1 is known (e.g., from spectroscopic type) and i is known (e.g., from eclipses), then one can actually estimate m_2 rather than just placing a lower limit. The point is that by observing one object in a binary, one can constrain the mass of the other. This is how the masses of extrasolar planets are estimated; by looking at the spectral lines of a star, one can often see periodic variation that indicates an orbit. From the mass of the star, Q, and P, one can then get $m \sin i$ for the planet. Binary observations have also provided the best evidence for stellar-mass black holes. In some systems, one can see a star orbiting around some X-ray emitting thing. In principle this could be a neutron star or a black hole, but in ~ 10 cases the mass function exceeds the upper limit for the mass of a neutron star, and hence a black hole is inferred.