Lecture 15: Cosmological Models

- Standard cosmological models - 3 types
- Hubble time and other terminology
- The Friedmann equation
- The Critical Density and $\Omega$

Reading for lectures 15 & 16: Chapter 11

The FRW metric

- According to GR, the possible space-time intervals in a homogeneous, isotropic Universe are the FRW metric forms with $k=0$ (flat), $k=1$ (spherical), $k=-1$ (hyperbolic):

$$\Delta s^2 = (c\Delta t)^2 - R(t)^2 \left( \frac{(\Delta r)^2}{1 - kr^2} + (\Delta \theta)^2 + \sin^2 \theta (\Delta \phi)^2 \right)$$

- The scale factor $R(t)$ describes the relative expansion of space itself as a function of time.
- For $k=1$, maximum $r=1$; for $k=0$ or $k=-1$, $r=0$ to $\infty$
The Hubble parameter

- Both physical distances between galaxies and wavelengths of radiation vary proportional to $R(t)$:
  - $d(t)=D_{\text{comoving}} R(t)$
  - $\lambda(t)=\lambda_{\text{emitted}} R(t)/R(\text{emitted})$
- Observed redshift of radiation from distant source is related to scale factor at emission time ($t_{\text{em}}$) and present time ($t_0$) by
  $$1+z=\lambda(t_0)/\lambda(t_{\text{em}})=R(t_0)/R(t_{\text{em}})$$
- Hubble observed that Universe is currently expanding; expansion can be characterized by
  $$H=(dR/dt)/R=\Delta R/(R\times\Delta t)$$
- For nearby galaxies, $v=d\times H_0$, where the present value of the Hubble parameter is approximately
  $$H_0=70 \text{ km/s/Mpc}$$
But how does $R(t)$ (and $H$) change in time? And what is the value of the curvature $k$?

Need to solve Einstein’s equation!

$$G = \frac{8\pi G}{c^4} T$$

First we will discuss all of the possible types of solutions that could exist for Einstein’s equation.

Later we will discuss which solution or solutions appear to hold in real Universe, based on current observations.
STANDARD COSMOLOGICAL MODELS

We’ll start with the answer, and then explain it...

- In general Einstein’s equation relates geometry to dynamics
- That means curvature must relate to evolution
- It turns out that there are three possibilities...

Types of spaces

Spherical space (closed=finite; \( k=+1 \))
Flat spaces (open=∞; k=0)

Hyperbolic spaces (open=∞; k=-1)

**Important features of standard models...**

- All models begin with $R \rightarrow 0$ at a finite time in the past
- This time is known as the **BIG BANG**
- Space and time come into existence at this moment... there is no time before the big bang!
- The big bang happens everywhere in space... not at a point!
There is a connection between the geometry and the dynamics

- Closed \((k=+1)\) solutions for universe expand to maximum size then re-collapse
- Open \((k=-1)\) solutions for universe expand forever
- Flat \((k=0)\) solution for universe expands forever (but only just barely... almost grinds to a halt).

Hubble time

We can relate this to observations...

- Once the Hubble parameter has been determined accurately from observations, it gives very useful information about age and size of the expanding Universe...
- Recall Hubble parameter is ratio of rate of change of size of Universe to size of Universe:
  \[
  H = \frac{1}{R} \frac{\Delta R}{\Delta t} = \frac{1}{R} \frac{dR}{dt}
  \]
- If Universe were expanding at a constant rate, we would have \(\Delta R / \Delta t = \text{constant} \) and \( R(t) = t \times (\Delta R / \Delta t) \); then would have \( H = (\Delta R / \Delta t) / R = 1/t \)
- ie \( t_H = 1/H \) would be age of Universe since Big Bang
Hubble time for nonuniform expansion

Hubble time is $t_H = \frac{1}{H} = \frac{R}{(dR/dt)}$

Since rate of expansion varies, $t_H = 1/H$ gives an estimate of the age of the Universe.

This tends to overestimate the age of the Universe since the Big Bang compared to the actual age.

Terminology

- Hubble distance, $D = ct_H$ (distance that light travels in a Hubble time). This gives an approximate idea of the size of the observable Universe.
- Age of the Universe, $t_{age}$ (the amount of cosmic time since the big bang). In standard models, this is always less than the Hubble time.
- Look-back time, $t_{lb}$ (amount of cosmic time that passes between the emission of light by a certain galaxy and the observation of that light by us).
- Particle horizon (a sphere centered on the Earth with radius $ct_{age}$; i.e., the sphere defined by the distance that light can travel since the big bang). This gives the edge of the actual observable Universe.
Friedmann Equation

Where do the three types of evolutionary solutions come from?
- Back to Einstein’s eq....
  \[ G = \frac{8\pi G}{c^4} \]
- When we put the FRW metric in Einstein’s equation and go though the GR, we get the Friedmann Equation... this is what determines the dynamics of the Universe
  \[ \left( \frac{dR}{dt} \right)^2 = \frac{8\pi G}{3} \rho R^2 - kc^2 \]

What are the terms involved?
- \( G \) is Newton’s universal constant of gravitation
- \( \frac{dR}{dt} \) is the rate of change of the cosmic scale factor
- This is same as \( \frac{\Delta R}{\Delta t} \) for small changes in time
- In textbook, symbol for \( \frac{dR}{dt} \) is \( \dot{R} \) (pronounced “R-dot”)
- \( \rho \) is the total energy density \( \div c^2 \); this equals mass/volume for “matter-dominated” Universe
- \( k \) is the geometric curvature constant (\( \equiv +1, 0, -1 \))

If we divide Friedmann equation by \( R^2 \), we get:

\[ H^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{R^2} \]

Let’s examine this equation...
- \( H^2 \) must be positive... so the RHS of this equation must also be positive.
- Suppose density is zero (\( \rho = 0 \))
  - Then, we must have negative \( k \) (i.e., \( k = -1 \))
  - So, empty universes are open and expand forever
  - Flat and spherical Universes can only occur in presence of (enough) matter.
Critical density

Now, suppose the Universe is flat \((k=0)\)

Friedmann equation then gives

\[ H^2 = \frac{8\pi G}{3} \rho \]

So, this case occurs if the density \(\rho\) is exactly equal to the critical density:

\[ \rho_{\text{crit}} = \frac{3H^2}{8\pi G} \]

“Critical” density means “flat” solution for a given value of \(H\), which is the most easily observed parameter

In general, we can define the density parameter:

\[ \Omega = \frac{\rho}{\rho_{\text{crit}}} = \frac{8\pi G \rho}{3H^2} \]

Can now rewrite Friedmann’s equation, moving the curvature term to the other side of the equation and dividing by \(H^2\)

We get:

\[ \Omega = 1 + \frac{kc^2}{H^2R^2} \]
Omega in standard models

$\Omega = 1 + \frac{kc^2}{H^2R^2}$

Thus, within context of the standard model:
- $\Omega < 1$ if $k = -1$; then universe is hyperbolic and will expand forever
- $\Omega = 1$ if $k = 0$; then universe is flat and will (just manage to) expand forever
- $\Omega > 1$ if $k = +1$; then universe is spherical and will recollapse

Physical interpretation:
If there is more than a certain amount of matter in the universe ($\rho > \rho_{critical}$), the attractive nature of gravity will ensure that the Universe recollapses!

T-shirt version

DENSITY IS DESTINY
**Value of critical density**

- For present best-observed value of the Hubble constant, $H_0=70 \text{ km/s/Mpc}$, the critical density, $\rho_{\text{critical}}=\frac{3H_0^2}{8\pi G}$, is equal to $\rho_{\text{critical}}=10^{-26} \text{ kg/m}^3$; i.e. 6 H atoms/m$^3$.

- Compare to:
  - $\rho_{\text{water}}=1000 \text{ kg/m}^3$
  - $\rho_{\text{air}}=1.25 \text{ kg/m}^3$ (at sea level)
  - $\rho_{\text{interstellar gas}}=2 \times 10^{-21} \text{ kg/m}^3$

*Planck finds $H_0=68 \text{ km/s/Mpc}*

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**Next time...**

- Deceleration parameter
- Beyond the standard models
- Cosmological constant models
- Solutions for special cases