Lecture 16: Cosmological Models II

- Deceleration parameter
- Beyond standard cosmological models
  - The Friedman equation with $\Lambda$
  - Effects of nonzero $\Lambda$
- Solutions for special cases
  - de Sitter solution
  - Static model
  - Steady state model

Recap

- The **Friedmann equation** is obtained by plugging each of the possible FRW metric cases into Einstein’s GR equation
- Result is an equation saying how the cosmic scale factor $R(t)$ must change in time:

\[
\left( \frac{dR}{dt} \right)^2 = H^2 R^2 = \frac{8\pi G}{3} \rho R^2 - k c^2
\]

- The Friedmann equation can also be written as:

\[
\Omega = 1 + \frac{k c^2}{H^2 R^2}
\]

- where

\[
\Omega \equiv \frac{\rho}{\rho_{\text{crit}}} \equiv \frac{\rho}{(3H^2 / 8\pi G)}
\]
Recap

- Whether case with $k = -1, 0, \text{ or } 1$ applies depends on the ratio of the actual density to the “critical” density, $\Omega$.
- Properties of standard model solutions:
  - $k = -1$, $\Omega < 1$ expands forever
  - $k = 0$, $\Omega = 1$ “just barely” expands forever
  - $k = +1$, $\Omega > 1$ expands to a maximum radius and then recollapses

The deceleration parameter, $q$

- The deceleration parameter measures how quickly the universe is decelerating (or accelerating), i.e. how much $R(t)$ graph curves.
- In standard models, deceleration occurs because the gravity of matter slows the rate of expansion.
- For those comfortable with calculus, actual definition of $q$ is:
  \[ q = -\frac{1}{RH^2} \frac{d^2 R}{dt^2} \]
  In the textbook, $d^2 R/dt^2$ is written as $\ddot{R}$, pronounced “$R$ double-dot”
Matter-only standard model

In standard model where density $\rho$ is entirely from the rest mass energy of matter, it turns out that the value of the deceleration parameter is given by

$$q = \frac{\Omega}{2}$$

This gives a consistency check for the standard, matter-dominated models... we can attempt to measure $\Omega$ in two ways:

- Direct measurement of how much mass is in the Universe ... i.e.
  measure mass density $\rho$, measure Hubble parameter $H$, and then compare $\rho$ to the critical value $\rho_{\text{crit}} = 3H^2/(8\pi G)$
- Use measurement of deceleration parameter, $q$

Measurement of $q$ is analogous to measurement of Hubble parameter, by observing change in expansion rate as a function of time: need to look at how $H$ changes with redshift for distant galaxies

Direct observation of $q$

Deceleration shows up as a deviation from Hubble’s law...

A very subtle effect - have to detect deviations from Hubble’s law for objects with a large redshift
Obtaining Friedmann eq. for mass-only Universe from Newtonian theory

- Consider spherical piece of the Universe large enough to contain many galaxies, but much smaller than the Hubble radius (distance light travels in a Hubble time).
- Radius is \( r \). Mass is \( M( r) = \frac{4\pi r^3 \rho}{3} \)
- Consider particle \( m \) at edge of this sphere; it feels gravitational force from interior of sphere, \( F = -GM(r)\frac{m}{r^2} \)
- Suppose outer edge, including \( m \), is expanding at a speed \( v(r) = \frac{\Delta r}{\Delta t} = \frac{dr}{dt} \)
- Then, from Newton’s 2nd law, rate of change of \( v \) is the acceleration \( a = \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \), with \( F = ma \), yielding \( a = \frac{F}{m} = -\frac{GM(r)}{r^2} = -\frac{4\pi G r \rho}{3} \)
- Using calculus, this can be worked on to obtain \( v^2 = \frac{8\pi G \rho r^2}{3} + \text{constant} \)

- But if we simply identify the “constant” (which is twice the Newtonian energy) with \(-kc^2\), and reinterpret \( r \) as \( R \), the cosmic scale factor, we have Friedmann’s equation!

\[
\left( \frac{dR}{dt} \right)^2 = \frac{8\pi G}{3} \rho R^2 - kC^2
\]
Newtonian interpretation is therefore:

- $k=-1$ is “positive energy” universe (which is why it expands forever)
- $k=+1$ is “negative energy” universe (which is why it recollapses at finite time)
- $k=0$ is “zero energy” universe (which is why it expands forever but slowly grinds to a halt at infinite time)

Analogies in terms of throwing a ball in the air...

Expansion rates

- For flat ($k=0$, $\Omega=1$), matter-dominated universe, it turns out there is a simple solution to how $R$ varies with $t$:

$$R(t) = R(t_0) \left( \frac{t}{t_0} \right)^{2/3}$$

- This is known as the Einstein-de Sitter solution
- For this solution,
  $$V = \Delta R/\Delta t = (2/3)(R(t_0)/t_0)(t/t_0)^{1/3}$$
  + How does this behave for large time? What is $H=V/R$?

- In solutions with $\Omega>1$, expansion is slower (followed by recollapse)
- In solutions with $\Omega<1$, expansion is faster
Open, flat, and closed solutions result for different values of $\Omega$.

- Open (hyperbolic) expands forever when $\Omega < 1$ and $k = -1$.
- Open (flat) just manages to expand forever when $\Omega = 1$ and $k = 0$.
- Closed (spherical) recollapse when $\Omega > 1$ and $k = +1$.

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**Modified Einstein’s equation**

- But Einstein’s equations most generally also can include an extra constant term; i.e. in
  
  $$G = \frac{8\pi G}{c^4} T$$

  the $T$ term has an additional term which just depends on space-time geometry times a constant factor, $\Lambda$.
- This constant $\Lambda$ (Greek letter “Lambda”) is known as the “cosmological constant”.
- $\Lambda$ corresponds to a “vacuum energy”, i.e. an energy not associated with either matter or radiation.
- $\Lambda$ could be positive or negative.
  - Positive $\Lambda$ would act as a repulsive force which tends to make the Universe expand faster.
  - Negative $\Lambda$ would act as an attractive force which tends to make the Universe expand slower.
- Energy terms in cosmology arising from positive $\Lambda$ are now often referred to as “dark energy.”
**Modified Friedmann Equation**

- When Einstein equation is modified to include $\Lambda$, the Friedmann equation governing evolution of $R(t)$ changes to become:

$$\left(\frac{dR}{dt}\right)^2 = H^2 R^2 = \frac{8\pi G}{3} \rho R^2 + \frac{\Lambda R^2}{3} - kc^2$$

- Dividing by $(HR)^2$, we can consider the relative contributions of the various terms evaluated at the present time, $t_0$.
- The term from matter at $t_0$ has subscript “M”;
- Two additional “$\Omega$” density parameter terms at $t_0$ are defined:

$$\Omega_M = \frac{\rho_0}{\rho_{\text{crit}}} = \frac{\rho_0}{(3H_0^2 / 8\pi G)} \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2} \quad \Omega_k = -\frac{k c^2}{R_0^2 H_0^2}$$

- Altogether, at the present time, $t_0$, we have

$$1 = \Omega_M + \Omega_\Lambda + \Omega_k$$

**Generalized Friedmann Equation in terms of $\Omega$’s**

- The generalized Friedmann equation governing evolution of $R(t)$ is written in terms of the present $\Omega$’s (density parameter terms) as:

$$\ddot{R}^2 = \left(\frac{dR}{dt}\right)^2 = H^2 R^2 = H_0^2 R_0^2 \left[ \Omega_M \left(\frac{R_0}{R}\right) + \Omega_\Lambda \left(\frac{R}{R_0}\right)^2 + \Omega_k \right]$$

- The only terms in this equation that vary with time are the scale factor $R$ and its rate of change $dR/dt$.
- Once the constants $H_0$, $\Omega_M$, $\Omega_\Lambda$, $\Omega_k$ are measured empirically (using observations), then whole future of the Universe is determined by solving this equation!
- Solutions, however, are more complicated than when $\Lambda=0$...
Effects of $\Lambda$

- Deceleration parameter (observable) now depends on both matter content and $\Lambda$ (will discuss more later)
- This changes the relation between evolution and geometry. Depending on value of $\Lambda$,
  - closed ($k=+1$) Universe could expand forever
  - flat ($k=0$) or hyperbolic ($k=-1$) Universe could recollapse

Consequences of positive $\Lambda$

- Because $\Lambda$ term appears with positive power of $R$ in Friedmann equation, effects of $\Lambda$ increase with time if $R$ keeps increasing
- Positive $\Lambda$ can create accelerating expansion!

$$\dot{R}^2 = \left(\frac{dR}{dt}\right)^2 = H^2 R^2 = H_0^2 R_0^2 \left[ \Omega_M \left(\frac{R_0}{R}\right) + \Omega_\Lambda \left(\frac{R}{R_0}\right)^2 + \Omega_k \right]$$
Special solutions

- “de Sitter” model:
  - Case with $\Omega_k = 0$ (flat space!), $\Omega_M = 0$ (no matter!), and $\Lambda > 0$
  - Then modified Friedmann equation reduces to
    \[
    \dot{R}^2 = H^2 R^2 = H_0^2 R_0^2 \Omega_\Lambda \left( \frac{R}{R_0} \right)^2 = \frac{R^2 \Lambda}{3}
    \]
  - Hubble parameter is constant:
    \[
    H = \frac{\dot{R}}{R} = \sqrt[3]{\frac{\Lambda}{3}}
    \]
  - Expansion is exponential:
    \[
    R = R_0 e^{H t / t_0}
    \]

- Static model (Einstein’s)
  - Solution with $\Lambda = 4 \pi G \rho$, $k = \Lambda R^2 / c^2$
  - No expansion: $H = 0$, $R = \text{constant}$
  - Closed (spherical)
  - Of historical interest only since Hubble’s discovery that Universe is expanding!
Other ideas

Steady solution:
- Constant expansion rate
- Matter constantly created
- No Big Bang

Ruled out by existing observations:
- Distant galaxies (seen as they were light travel time in the past) differ from modern galaxies
- Cosmic microwave background implies earlier state with uniform hot conditions (big bang)
- Observed deceleration parameter differs from what would be required for steady model

“asymptotic” behavior

\[ \dot{R}^2 = \left( \frac{dR}{dt} \right)^2 = H^2 R^2 = H_0^2 R_0^2 \left[ \Omega_M \left( \frac{R_0}{R} \right) + \Omega_\Lambda \left( \frac{R}{R_0} \right)^2 + \Omega_k \right] \]

Different terms in modified Friedmann equation are important at different times...
- Early times \( \Rightarrow R \) is small
- Late times \( \Rightarrow R \) is large

When can curvature term be neglected?
- When can lambda term be neglected?
- When can matter term be neglected?
- How does \( R \) depend on \( t \) at early times in all solutions?
- How does \( R \) depend on \( t \) at late times in all solutions?
- What is the ultimate fate of the Universe if \( \Lambda \neq 0 \)?
Next lecture...

- The Early Universe
  - Cosmic radiation and matter densities
  - The hot big bang
  - Fundamental particles and forces
  - Stages of evolution in the early Universe