Class 14. Ordinary Differential Equations

- *NRiC* §16.
- ODEs involve derivatives with respect to *one* independent variable, e.g., time t.
- ODEs can always be reduced to a *set* of first-order equations (i.e., involving *only* first derivatives). E.g.,

$$\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} = c(t)$$

  is equivalent to the set

$$\begin{aligned} \frac{dy}{dt} &= z(t), \\ \frac{dz}{dt} &= c(t) - b(t)z(t). \end{aligned}$$

  - Example: gravity! In 1-D,

$$F = ma = m\ddot{x} = -\frac{GMm}{x^2} = F_g.$$

    Let $z(t) = \dot{x}$. Then $\dot{z} = -GM/x^2$. In 3-D, just write out equations for each component (we'll see this again...).

- Usually new variables just derivatives of old, but sometimes need additional factors of t to avoid pathologies.
- General problem is solving set of 1^{st}-order ODEs,

$$\frac{dy_i}{dt} = f_i'(t, y_1, ..., y_N),$$

  where the f_i' are known functions.[1]

- But, also need <u>boundary conditions</u>: algebraic conditions on values of y_i at discrete time(s) t...

[1] Often ODEs are coupled to begin with, e.g., classic Lotka-Volterra predator-prey model:

$$\begin{aligned} \dot{x} &= Ax - Bxy - ex, \\ \dot{y} &= -Cy + Dxy - dy. \end{aligned}$$

Here x and y might represent the population of rabbits and foxes, respectively. Then A is the reproduction rate of the rabbits, B is the consumption rate of rabbits by the foxes, C is the death rate by natural causes of the foxes, and D is the population increase rate of the foxes due to consumption of rabbits. We've also added terms with coefficients d and e representing the hunting rate by humans. For $d = e = 0$, the equilibrium solution of this system is cyclical.

ODE Boundary Conditions (BCs)

- Two categories of BC:

  1. Initial Value Problem (IVP): all y_i's are given at some starting point t_s, and solution is needed from t_s to t_f.
  2. Two-point Boundary Value Problem (BVP): y_i are specified at two or more t, e.g., some at t_s, some at t_f (only one BC needed for each y_i).

- Generally, IVP much easier to solve than 2-pt BVP, so consider this first.

Finite Differences

- How do you represent derivatives with a discrete number system?

- Basic idea: replace dy/dt with finite differences $\Delta y/\Delta t$. Then:

$$\lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \to \frac{dy}{dt}.$$

- How do you use this to solve ODEs?

Euler's Method

- Write $\Delta \mathbf{y}/\Delta t = \mathbf{f}'(t, \mathbf{y}) \Rightarrow \Delta \mathbf{y} = \Delta t\, \mathbf{f}'(t, \mathbf{y})$.

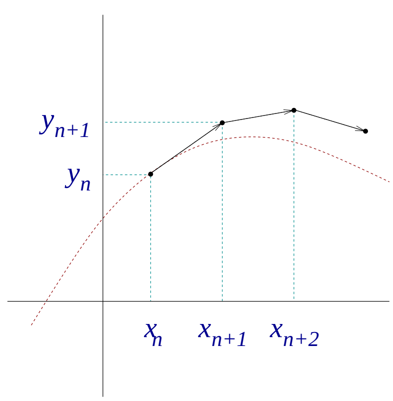
- Start with known values $\mathbf{y}_n$ at t_n (initial values).

- Then $\mathbf{y}_{n+1}$ at $t_{n+1} = t_n + h$ is

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}'(t_n, \mathbf{y}_n).$$

- h is called the *step size*.

- Integration is not symmetric: derivative evaluated only at start of step $\Rightarrow$ error term $\mathcal{O}(h^2)$, from Taylor series ($f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + ...$). So, Euler's method is first order.



- Example: consider $\dot{y} = y$ with $y(0) = 1$. We know the solution to be $y = e^t$. Using Euler's method with $h = 1/2$, we find

$$\begin{array}{lllll} y_0 & = & 1, & & \\ y_1 & = & y_0 + y_0/2 & = & 3/2, \\ y_2 & = & y_1 + y_1/2 & = & 9/4, \\ y_3 & = & y_2 + y_2/2 & = & 27/8, \\ \vdots & & \vdots & & \vdots \\ y_n & = & (\frac{3}{2})^n, & & \end{array}$$

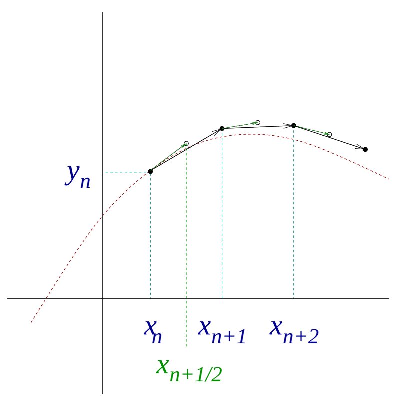
i.e., the solution is always $\leq e^t$ (since $t = nh = n/2$ and $e^{1/2} \doteq 1.65$).

Runge-Kutta Methods

- We can do better by symmetrizing the derivative:
  - Take a trial Euler step to midpoint: compute $t_{n+1/2}$ and evaluate $\mathbf{y}_{n+1/2}$.
  - Use these to evaluate derivative $\mathbf{f}'(t_{n+1/2}, \mathbf{y}_{n+1/2})$.
  - Then use this to go back and take a full step.
- Thus:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}'\left[t_n + \frac{1}{2}h, \mathbf{y}_n + \frac{1}{2}h\mathbf{f}'(t_n, \mathbf{y}_n)\right] + \mathcal{O}(h^3).$$

- Can show that $\mathcal{O}(h^2)$ terms "cancel," so leading error term is $\mathcal{O}(h^3)$, giving 2nd-order Runge-Kutta (midpoint method).



- Following previous example, first step using midpoint method:

$$\begin{array}{lll} y_1 & = & y_0 + (1/2)f'(0 + 1/4, 1 + (1/4)f'(0,1)), \\ & = & 1 + (1/2)f'(1/4, 5/4), \\ & = & 1 + (1/2)(5/4), \\ & = & 1 + 5/8, \\ & = & 1.625. \end{array}$$

- The idea behind midpoint method is to use Euler but with derivative at midpoint:

$$y(t) + hf'(t + \frac{1}{2}h) = y(t) + h\left[f'(t) + \frac{1}{2}hf'(t)\right] + \mathcal{O}(h^3).$$

  This is essentially a Taylor series within a Taylor series.
- Use Euler to determine derivative at midpoint:

$$\begin{aligned} k_1 &= hf'(t_n, y_n), \\ k_2 &= hf'(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1), \\ y_{n+1} &= y_n + k_2 + \mathcal{O}(h^3). \end{aligned}$$

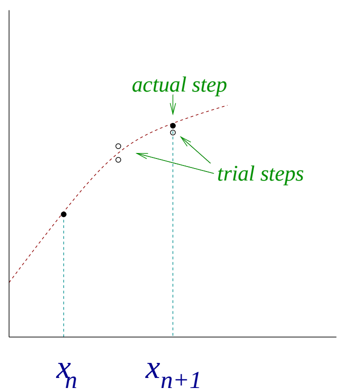
Fourth-order Runge-Kutta

- Actually, there are many ways to evaluate $\mathbf{f}'$ at midpoints, which add higher-order error terms with different coefficients. Can add these together in ways such that higher-order error terms cancel. E.g., can build 4th-order Runge-Kutta (RK4):

$$\begin{aligned} \mathbf{k}_1 &= h\mathbf{f}'(t_n, \mathbf{y}_n), \\ \mathbf{k}_2 &= h\mathbf{f}'(t_n + h/2, \mathbf{y}_n + \mathbf{k}_1/2), \\ \mathbf{k}_3 &= h\mathbf{f}'(t_n + h/2, \mathbf{y}_n + \mathbf{k}_2/2), \\ \mathbf{k}_4 &= h\mathbf{f}'(t_n + h, \mathbf{y}_n + \mathbf{k}_3). \end{aligned}$$

  Then:

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \mathbf{k}_1/6 + \mathbf{k}_2/3 + \mathbf{k}_3/3 + \mathbf{k}_4/6 + \mathcal{O}(h^5).$$



- Disadvantage of RK4: requires $\mathbf{f}'$ to be evaluated 4 times per step.
- But, can still be cost effective if larger steps OK.
- RK4 is a workhorse method. Higher-order RK4 takes too much effort for increased accuracy.
- Other methods (e.g., Bulirsch-Stoer, *NRiC* §16.4) are more accurate for smooth functions.
- But RK4 often "good enough."