Class 23. PDEs, Part 2

Solving Hyperbolic PDEs, Continued

Upwind differencing

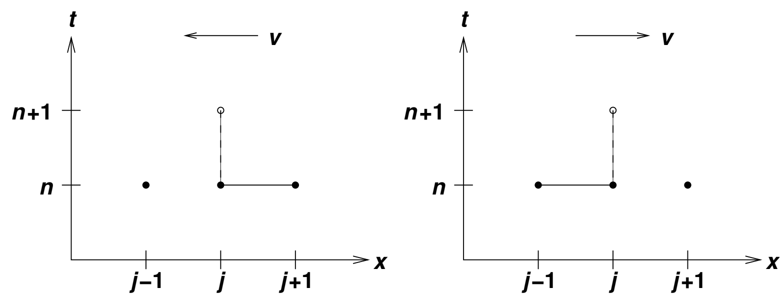
- In addition to *amplitude* errors (instability or damping), scheme may also have *phase* errors (dispersion) or *transport* errors (spurious transport of information).

- *Upwind differencing* helps reduce transport errors:

$$\frac{u_j^{n+1} - u_j}{\Delta t} = -v_j^n \begin{cases} \frac{u_j^n - u_{j-1}^n}{\Delta x}, & v_j^n > 0, \\ \frac{u_{j+1}^n - u_j^n}{\Delta x}, & v_j^n < 0, \end{cases}$$

  where here we've supposed that v is not constant, for illustration.

- Schematically, only use information upwind of grid point j to construct differences:



- Upwind difference is only first order in space. Still, it has lower transport error than second-order centered difference. Better? Can construct higher-order upwind difference schemes...

Second-order accuracy in time

- We have been dealing with two derivatives, $\partial/\partial x$ *and* $\partial/\partial t$. We have constructed higher-order schemes in space. What about t?

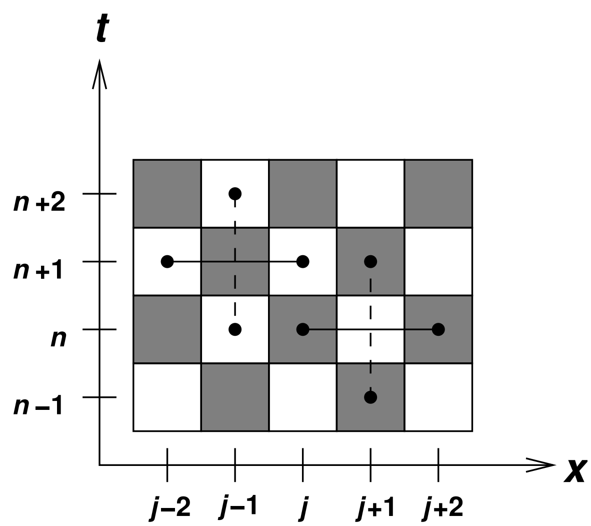
- Staggered leapfrog is 2^{nd}-order in time:

$$\frac{\partial}{\partial t} \to \frac{u_j^{n+1} - u_j^{n-1}}{\Delta t} = -\left(\frac{F_{j+1}^n - F_{j-1}^n}{\Delta x}\right).$$

  But, subject to a *mesh-drift* instability. Think of space-time discretization:

  - Odd-integer n coupled to even-integer j,
  - Even-integer n coupled to odd-integer j

  ("red-black" ordering; odd and even mesh points decoupled). Schematically,



Can be fixed by adding diffusion to couple grid points (add $\epsilon(F^n_{j-1} - 2F^n_j + F^n_{j+1})$, $\epsilon \ll 1$ to RHS).

- Two-step Lax-Wendroff: another $2^{\rm nd}$-order scheme.

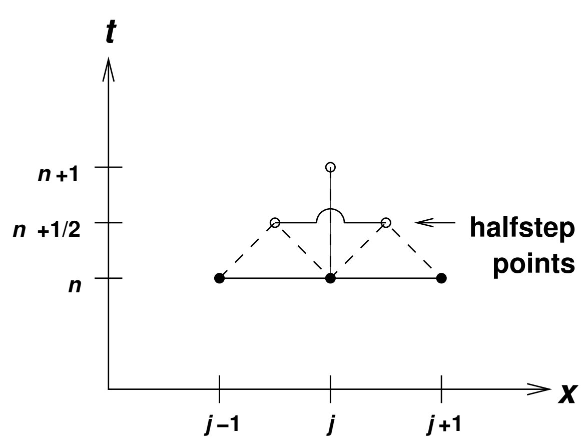
  1. Use Lax step to estimate fluxes at $n + \frac{1}{2}$ and $j \pm \frac{1}{2}$:

$$
\begin{aligned}
u^{n+1/2}_{j-1/2} &= \frac{u^n_{j-1} + u^n_j}{2} - \frac{\Delta t}{2\Delta x}\left(F^n_j - F^n_{j-1}\right), \\
u^{n+1/2}_{j+1/2} &= \frac{u^n_j + u^n_{j+1}}{2} - \frac{\Delta t}{2\Delta x}\left(F^n_{j+1} - F^n_j\right).
\end{aligned}
$$

  2. Using these half-step values of u, calculate $F(u^{n+1/2}_{j\pm 1/2}) \equiv F^{n+1/2}_{j\pm 1/2}$.
  3. Then use leapfrog to get updated values:

$$
u^{n+1}_j = u^n_j - \frac{\Delta t}{\Delta x}\left(F^{n+1/2}_{j+1/2} - F^{n+1/2}_{j-1/2}\right).
$$

Schematically,



Fixes dissipation and mesh drifting but introduces phase error (dispersion). Often first-order upwind scheme is as good as/better than $2^{\rm nd}$-order L-W.

Summary: Hyperbolic methods

- Many IVPs can be cast in flux-conservative form.
- Solving methods:
  1. FTCS — unconditionally unstable. Never use.
  2. Lax — equivalent to adding diffusion, damps small scales.
  3. Upwind differencing — reduces transport errors, but only 1^{st}-order in space.
  4. Staggered leapfrog — 2^{nd}-order in time, but subject to mesh-drift instability. Fix with diffusion.
  5. Two-step Lax-Wendroff — 2^{nd}-order in time, but suffers from phase error.
- *NRiC* recommends staggered leapfrog (presumably with diffusion), particularly for problems related to the wave equation.
- For problems sensitive to transport errors, *NRiC* recommends upwind differencing schemes.

Solving Parabolic PDEs (Diffusive IVPs)

- *NRiC* §19.2.
- Prototypical parabolic PDE is diffusion equation:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2},$$

  where we have taken $D > 0$ to be constant ($D = 0$ is trivial and $D < 0$ leads to physically unstable solutions).
- Consider FTCS differencing:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\left[\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{(\Delta x)^2}\right].$$

- von Neumann analysis gives (E.F.T.S.)

$$\xi(k) = 1 - \frac{4D\Delta t}{(\Delta x)^2}\sin^2\left(\frac{k\Delta x}{2}\right).$$

  This is stable provided (E.F.T.S.)

$$\frac{2D\Delta t}{(\Delta x)^2} \le 1.$$

  The 2^{nd} derivative makes all the difference (we saw adding diffusion via the Lax method stabilizes FTCS for the hyperbolic equation).

- Diffusion time over scale L is $\tau_D \sim L^2/D$. So stability criterion says $\Delta t \lesssim \tau_D/2$ across one cell.
- Often interested in evolution of time scales $\gg \tau_D$ of one cell. How can we build stable scheme for larger Δt?

Implicit differencing

- Evaluate RHS of difference equation at $n+1$:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\left[\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{(\Delta x)^2}\right].$$

- To solve this, rewrite as:

$$-\alpha u_{j-1}^{n+1} + (1+2\alpha)u_j^{n+1} - \alpha u_{j+1}^{n+1} = u_j^n, \tag{1}$$

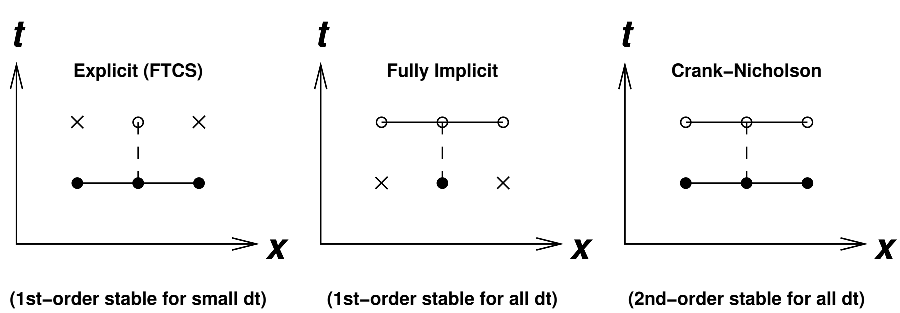
  where $\alpha \equiv D\Delta t/(\Delta x)^2$.
  - In 1-D, this is a tri-di matrix.
  - In 3-D, get large, sparse, banded matrix.
  - Solve the usual way.
- What is limit of (1) as $\Delta t \to \infty$ ($\alpha \to \infty$)? Divide through by α to find FD form of $\partial^2 u/\partial x^2 = 0$, i.e., static solution.
- Fully implicit scheme is unconditionally stable (E.F.T.S.) and gives correct equilibrium structure, but cannot be used to follow small-timescale phenomena.

Crank-Nicholson differencing

- Form average of explicit and implicit schemes (in space):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = D\left[\frac{(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + (u_{j-1}^{n} - 2u_j^{n} + u_{j-1}^{n})}{2(\Delta x)^2}\right].$$

- Unconditionally stable (E.F.T.S.), 2nd-order accurate in time (both sides centered at $n+1/2$).
- Schematically,



- “Freezes” small-scale phenomena. Can use fully implicit scheme at end to drive fluctuations to equilibrium.

Nonlinear diffusion problems

- For nonlinear diffusion problems, e.g., where $D = D(x)$, then implicit differencing more complex.
- Must linearize system and use iterative methods.