Class 24. Fluid Dynamics, Part 1

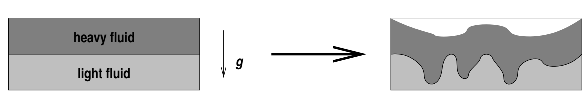
- The equations of fluid dynamics are coupled PDEs that form an IVP (hyperbolic).
- Use the techniques described so far, plus additions.

Fluid Dynamics in Astrophysics

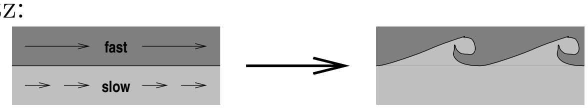
- Whenever mean free path $\lambda \ll$ problem scale L in a plasma, can use continuum equations to describe evolution of macroscopic variables, e.g., density, pressure, etc.
- Mathematically,
$$\lambda \simeq \frac{1}{\sigma n} \sim \frac{10^{16}}{[n/1\ \mathrm{cm}^{-3}]}\ \mathrm{cm},$$
where σ = classical cross-section of atom or ion ($\sim \pi r_{\mathrm{Bohr}}^2$).
- Where is $\lambda \ll L$ in astrophysics?

Medium	$\sim n$ (cm^{-3})	$\sim \lambda$ (cm)	$\sim L$ (cm)	Scale
planetary atmosphere	10^{20}	10^{-4}	10^{2-3}	1–10 m
stellar interior	10^{24}	10^{-8}	10^{11}	1 $R_\odot$
protoplanetary disk	10^{10}	10^{6}	10^{13}	1 AU
GMC	10	10^{15}	10^{19}	10 pc
diffuse ISM	1	10^{16}	10^{20}	100 pc
cluster gas	0.1	10^{17}	10^{22}	10 kpc
universe	10^{-6}	10^{22}	$> 10^{24}$	> 1 Mpc

- What would we like to learn from studying fluid dynamics?
  1. Steady-state structure of certain fluid flows, e.g., C-shocks ("continuous").
  2. Time evolution of system, e.g.,
     - Propagation of shock through clumpy medium.
     - Accretion flow onto protostar or black hole.
     - Formation of structure in universe.
  3. Growth and saturation of instabilities, e.g.,
     - Rayleigh-Taylor:



       * Important in SN explosions, ISM, etc.
     - Kelvin-Helmholtz:



       * Important in jets and outflows in ISM.
- To study these phenomena, must use equations of fluid dynamics.

Equations of Fluid Dynamics

1. Continuity equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \mathbf{v}) = 0, \tag{1}$$

where ρ = mass density, $\mathbf{v}$ = velocity, and $\boldsymbol{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$.

- Sometimes see this written as:

$$\frac{D\rho}{Dt} = -\rho \boldsymbol{\nabla}\cdot\mathbf{v},$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}$ = Lagrangian or co-moving or substantive derivative (rate of change of ρ in fluid frame, as opposed to $\frac{\partial}{\partial t}$ = Eulerian derivative, rate of change in lab frame).

- For an incompressible fluid, ρ is constant in space and time, so the continuity equation reduces to:

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0.$$

- The continuity equation is a statement of *mass conservation.*

2. Euler's equation (equation of motion):

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = \frac{\mathbf{F}}{\rho} - \frac{1}{\rho}\boldsymbol{\nabla} p, \tag{2}$$

where p = pressure and $\mathbf{F}$ = any external force (other than gas pressure) acting on a unit volume.

- More compactly,

$$\rho \frac{D\mathbf{v}}{Dt} = \mathbf{F} - \boldsymbol{\nabla} p.$$

- For gravity, have $\mathbf{F} = -\rho \boldsymbol{\nabla}\phi$, where $\boldsymbol{\nabla}^2 \phi = 4\pi G \rho$. In hydrostatic equilibrium, $\mathbf{F} = \boldsymbol{\nabla} p$, so there is no mass flow. E.g., in 1-D, have $dp/dr = -\rho\, GM(r)/r^2 = -g\rho$, where g = gravitational acceleration.
- For viscosity, $\mathbf{F} = \mu \boldsymbol{\nabla}^2 \mathbf{v}$, where μ = coefficient of dynamical viscosity, assuming ρ = constant (incompressible fluid). If there are no other force terms in $\mathbf{F}$, this gives the Navier-Stokes equation.
- Similarly, can add force terms for electric and/or magnetic fields.
- For the steady flow of a gas, $\partial \mathbf{v}/\partial t = \mathbf{0}$ and, if there are no external forces, get

$$\rho\, \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\boldsymbol{\nabla} p,$$

which is Bernoulli's equation for compressible flow.

- Euler's equation is a statement of *momentum conservation.*

3. Energy equation:

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla\cdot} [(e+p)\mathbf{v}] = 0, \tag{3}$$

where $e \equiv \rho(\varepsilon + \frac{1}{2}v^2)$ = energy density (energy/volume) and ε = specific internal energy (energy/mass).

- In Lagrange form,

$$\frac{De}{Dt} = -e(\boldsymbol{\nabla\cdot}\mathbf{v}) - \boldsymbol{\nabla\cdot}(p\mathbf{v}),$$

or, more compactly,

$$\frac{D\varepsilon}{Dt} = -\frac{p}{\rho}(\boldsymbol{\nabla\cdot}\mathbf{v}).$$

- The energy equation is a statement of *energy conservation* (there are many alternative ways to write the energy equation, depending on the context, e.g., using specific enthalpy ($= \varepsilon + p/\rho$), specific entropy combined with temperature and heat transfer, etc.).

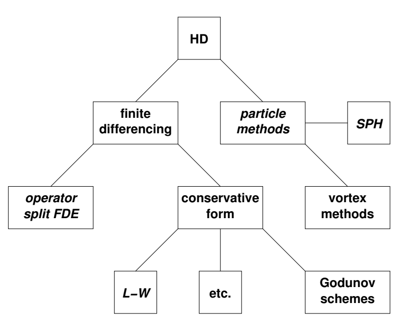
4. Equation of state:

$$p = p(\rho, \varepsilon). \tag{4}$$

- Needed to close system.
- E.g., for ideal gas, $p = (\gamma - 1)\rho\varepsilon$, where γ = adiabatic index (= ratio of specific heats at constant volume and pressure).[1] For ideal monatomic, diatomic, and polyatomic gases, $\gamma = 5/3$, $7/5$, and $4/3$, respectively.

Solving the Equations of Fluid Dynamics

- There are many choices one can make when adopting a numerical algorithm to solve the equations of fluid dynamics, e.g.,
  1. Finite differencing methods, including:
     (a) Flux-conservative form.
     (b) Operator splitting.
  2. Particle methods (e.g., smoothed particle hydrodynamics, or SPH).
- Schematically (will discuss methods in *italics*),



[1]Also have pV^γ = constant, $TV^{\gamma-1}$ = constant, $Tp^{(1-\gamma)/\gamma}$ = constant.