Class 26. Fourier Transforms, Part 1

Introduction

- Cf. *NRiC* §12.0.
- Fourier theorem: a well-behaved function can be represented by a series of sines and cosines of different frequencies and amplitudes.
- Often useful to know what these frequencies and amplitudes are. Can do this with a *Fourier transform*:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt,$$

where $-\infty < f < \infty$ is the frequency and $H(f)$ is the amplitude (H is often complex, i.e., contains phase info).

- *Inverse* Fourier transform:

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{-2\pi i f t} df.$$

- Units: if t is in seconds, f is in Hertz. If have $h(x)$, x in m, then get $H(n)$, $n =$ wavenumber (m^{-1}).
- FTs are linear ops:

$$\begin{aligned} \mathrm{FT}(g+h) &= \mathrm{FT}(g) + \mathrm{FT}(h), \\ \mathrm{FT}(\alpha h) &= \alpha \mathrm{FT}(h). \end{aligned}$$

- $h(t)$ may have special symmetries, e.g., pure real or pure imaginary, even ($h(t) = h(-t)$) or odd ($h(t) = -h(-t)$) $\Longrightarrow$ can increase computational efficiency:

$$\begin{aligned} h(t) \text{ pure real} &\Longrightarrow H(-f) = H^\star(f) \\ h(t) \text{ pure imaginary} &\Longrightarrow H(-f) = -H^\star(f) \\ h(t) \text{ real \& even} &\Longrightarrow H(f) \text{ real \& even} \\ h(t) \text{ real \& odd} &\Longrightarrow H(f) \text{ imaginary \& odd} \\ \text{etc.} & \end{aligned}$$

Other properties, and combinations

- If $h(t) \Longleftrightarrow H(f)$ are a FT pair, then

$$\begin{aligned} h(at) &\Longleftrightarrow \frac{1}{|a|} H(\tfrac{f}{a}) && \text{“time scaling”} \\ \frac{1}{|b|} h(\tfrac{t}{b}) &\Longleftrightarrow H(bf) && \text{“frequency scaling”} \\ h(t - t_0) &\Longleftrightarrow H(f) e^{2\pi i f t_0} && \text{“time shifting”} \\ h(t) e^{-2\pi i f_0 t} &\Longleftrightarrow H(f - f_0) && \text{“frequency shifting”} \end{aligned}$$

- Combinations: if $h(t) \Longleftrightarrow H(f)$ and $g(t) \Longleftrightarrow G(f)$, then
  1. Convolution:
  $$g \star h \equiv \int_{-\infty}^{\infty} g(\tau)h(t-\tau)\,d\tau.$$
     - Function of time. Note $g \star h = h \star g$.
  2. Convolution theorem:
  $$g \star h \Longleftrightarrow G(f)H(f)$$
     - E.g., instrumental profile (point spread function): observe star, get PSF (convolution of instrumental profile with delta function), now observe target, take FT, divide by FT of PSF, take inverse FT to get deconvolved image.
  3. Correlation:
  $$\mathrm{corr}(g,h) = \int_{-\infty}^{\infty} g(\tau+t)h(\tau)\,d\tau.$$
     - Function of time, called "lag."
     - Note $\mathrm{corr}(g,h) \Longleftrightarrow G(f)H(-f) = G(f)H^{\star}(f)$ if $h(t)$ real.
     - Correlation used to compare data sets: it's large at some t if functions are close copies of each other but lead or lag in time by t. E.g., Doppler shift!
  4. Wiener-Khinchin theorem (*autocorrelation*):
  $$\mathrm{corr}(g,g) \Longleftrightarrow |G(f)|^2.$$
  5. Parseval's theorem:
  $$\text{total power} = \int_{-\infty}^{\infty} |h(t)|^2\,dt = \int_{-\infty}^{\infty} |H(f)|^2\,df.$$
- Often interested in power between f and $f + df$. Usually regard f as varying from 0 (D.C.) to $+\infty \Longrightarrow$ one-sided *power spectral density* (PSD):
$$P_h(f) \equiv |H(f)|^2 + |H(-f)|^2,\ 0 \le f < \infty.$$
If $h(t)$ real, $P_h(f) = 2|H(f)|^2$.
- If $h(t)$ goes endlessly from $-\infty < t < \infty$, total power and PSD will generally be infinite. Instead compute PSD per unit time, i.e. PSD/sample length. Area then corresponds to mean square amplitude. As sample length $\to \infty$, PSD per unit time $\to$ delta functions for pure sines and cosines.

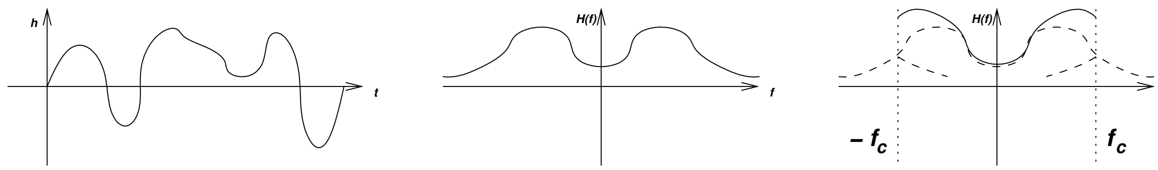
Discretely Sampled Data

- Cf. *NRiC* §12.1.
- For real data, often have $h_k \equiv h(t_k)$, $t_k = k\Delta$, $k = 0, 1, ..., N-1$. Here Δ is the sampling interval; $1/\Delta$ is the sampling rate.

- Define *Nyquist* critical frequency $f_c \equiv \frac{1}{2\Delta}$. Critical sampling of a sine wave of frequency f_c is two points per cycle.

- Sampling theorem: if signal is *bandwidth limited* such that $H(f) = 0$ for all $|f| \geq f_c$, then entire information content of signal can be recorded by sampling at $\Delta^{-1} = 2f_c$.

- If $h(t)$ has power in frequencies *outside* $-f_c < f < f_c$, sampling $h(t)$ causes power to spuriously move inside this range $\Longrightarrow$ *aliasing*:



  - Solution: filter signal and sample at least 2 points/cycle for highest frequency.

- If $h(t)$ finite in time, N points should sample entire interval. If $h(t)$ infinite, use representative portion.

- N inputs $\Longrightarrow$ N outputs:

$$f_n \equiv \frac{n}{N\Delta}, \qquad n = -\frac{N}{2}, ..., \frac{N}{2}.$$

  (For simplicity, assume N is even.) Extreme values of $n \Longleftrightarrow$ Nyquist frequency range.

- Now approximate:

$$H(f_n) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f_n t}\, dt \simeq \sum_{k=0}^{N-1} h_k e^{2\pi i f_n t_k} \Delta = \Delta \underbrace{\sum_{k=0}^{N-1} h_k e^{2\pi i k n/N}}_{\equiv H_n \text{ (DFT)}}.$$

- Note $H_{-n} = H_{N-n}$ if $n = 1, 2, ...$ (period N). Convention: let $n = 0, 1, ..., N-1$ so n and k vary over same range. $\therefore n = 0 \Longleftrightarrow$ zero frequency, $n = N/2 \Longleftrightarrow f = f_c$ *and* $f = -f_c$. Hence:

$$\begin{aligned} 1 \leq n \leq N/2 - 1 \quad &\Longleftrightarrow \quad 0 < f < f_c, \\ N/2 + 1 \leq n \leq N - 1 \quad &\Longleftrightarrow \quad -f_c < f < 0. \end{aligned}$$

  Also note $H(-f) \Longleftrightarrow H_{n-N}$.

- Discrete inverse Fourier transform:

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i k n/N}.$$

  Very similar to $H_n \Longrightarrow$ can use same code...

Application: Solving Poisson's Equation

- Cf. *NRiC* §19.4.
- Recall in 2-D the prototypical elliptic equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \rho(x, y).$$

- The FD version is (assuming $\Delta x = \Delta y \equiv \Delta$)

$$\frac{u_{j-1,k} - 2u_{j,k} + u_{j+1,k}}{\Delta^2} + \frac{u_{j,k-1} - 2u_{j,k} + u_{j,k+1}}{\Delta^2} = \rho_{j,k}. \quad (1)$$

- Consider letting $u_{j,k}$ be the 2-D inverse DFT of the Fourier-domain equivalent of u:

$$u_{j,k} = \frac{1}{JK} \sum_{m=0}^{J-1} \sum_{n=0}^{K-1} \hat{u}_{m,n} e^{-2\pi imj/J} e^{-2\pi ink/K}. \quad (2)$$

(In multi-D, FTs can be computed independently in each dimension.)

- Similarly,

$$\rho_{j,k} = \frac{1}{JK} \sum_{m=0}^{J-1} \sum_{n=0}^{K-1} \hat{\rho}_{m,n} e^{-2\pi imj/J} e^{-2\pi ink/K}. \quad (3)$$

- Substituting (2) and (3) into (1), we get

$$\hat{u}_{m,n} \left(e^{2\pi im/J} + e^{-2\pi im/J} + e^{2\pi in/K} + e^{-2\pi in/K} - 4 \right) = \hat{\rho}_{m,n} \Delta^2,$$

or

$$\hat{u}_{m,n} = \frac{\hat{\rho}_{m,n} \Delta^2}{2 \left(\cos \frac{2\pi m}{J} + \cos \frac{2\pi n}{K} - 2 \right)}. \quad (4)$$

- Strategy:
  1. Compute $\hat{\rho}_{m,n}$ as the FT

$$\hat{\rho}_{j,k} = \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} \rho_{j,k} e^{2\pi imj/J} e^{2\pi ink/K}.$$

  2. Compute $\hat{u}_{m,n}$ from (4).
  3. Compute $u_{j,k}$ by inverse FT (2).
- Procedure valid *only* for periodic boundary conditions, i.e., for $u_{j,k} = u_{j+J,k} = u_{j,k+K}$.
- All we need now is a fast way to compute the transforms!...