Data Representations

- Computers store data as different variable types, e.g. integer, floating point, complex, etc.
- Different machines have different wordlengths, e.g. 4-byte `int`s on a 32-bit machine (Pentium), 8-byte `int`s on a 64-bit machine (Alpha).
- This makes (binary) data non-portable.

Integers

- All data types represented by 0's and 1's.
- An integer value:

$$j = \sum_{i=1}^{N} s_i \times 2^{N-i}$$

  - N = # of bits in word
  - s_i = value of bit i in binary string s

- e.g. 0 0 0 0 0 1 1 0 = $2^2 + 2^1$ = 6 for 8-bit word.
- Use "two's complement" method for sign.

Integers, Cont'd

- Largest value that can be represented is 2^N - 1.
- For 32-bit word this is 4,294,967,295.
- Arithmetic with integers is exact, except:
  - When division results in remainder.
  - Result exceeds largest representable integer

    e.g. $2 \times 10^9 + 3 \times 10^9$ = overflow error
- Note multiplication by 2's can be achieved by left-shift, which is very fast (in C: "<<" operator).

Two's Complement

- Exploits finite size of data representations (cyclic groups) and properties of binary arithmetic.
- To get negative of binary number, invert all bits and add 1 to the result.

  e.g. 1 = 0 0 0 0 0 0 0 1 in 8-bit

invert bits:	1 1 1 1 1 1 1 0
add 1:	0 0 0 0 0 0 0 1
result:	1 1 1 1 1 1 1 1 = -1

- In 8 bits, signed char ranges from -128 to +127.

Negative Powers of 2

- Binary notation can be extended to cover negative powers of 2, e.g. "110.101" is:

  $1 \times 2^{2} + 1 \times 2^{1} + 1 \times 2^{-1} + 1 \times 2^{-3} = 6.625$

- Can represent real numbers by specifying some location in the word as the "binary point" → fixed-point representation.
- In practice, use some bits for an exponent → floating-point representation.

Floats

- For most machines these days, real numbers are represented by floating-point format:

$$x = s \times M \times B^{e-E}$$

s	= sign	B	= base (usually 2, sometimes 16)
M	= mantissa	e	= exponent
E	= bias, usually 127.		

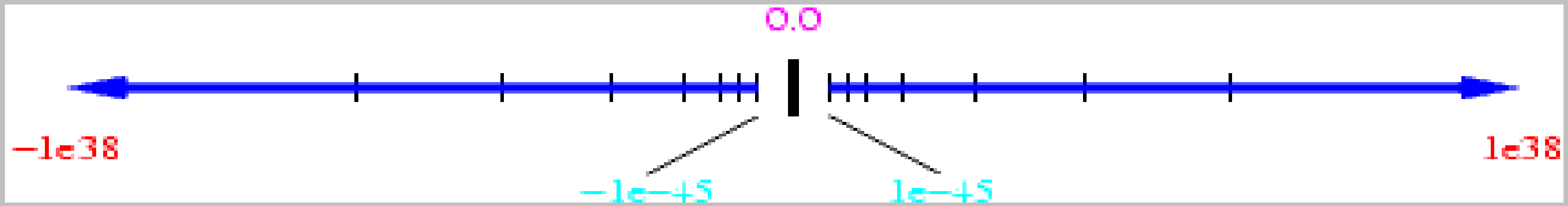
- In past, manufacturers used different number of bits for each of M and e $\rightarrow$ non-portable code.

Floats, Cont'd

- Currently, most manufacturers adopt IEEE standard:
  - s = 1st bit
  - Next 8 bits are e
  - Last 23 bits are M, expressed as a binary fraction, either 1.F, or, if e=0, 0.F, where F is in base 2.
- Largest single-precision float $f_{max} = 2^{127} \approx 10^{38}$.
- Smallest (and least precise!) $f_{min} = 2^{-149} \approx 10^{-45}$.

Round-off Error

- Not all values along real axis can be represented.
- There are 10^{38} integers between f_{min} & f_{max}, but only $2^{32} \approx 10^{9}$ bit patterns.



- Values < $|10^{-45}|$ result in "underflow" error.
- If value cannot be represented, next nearest value is produced. Difference between desired and actual value is called "round-off error" (RE).

Round-off Error, Cont'd

- Smallest value e_m for which $1 + e_m > 1$ is called "machine accuracy", typically $\sim 10^{-7}$ for 32 bits.
- Double precision greatly reduces e_m ($\sim 10^{-16}$).
- RE accumulates in a calculation:
  - Random walk: total error $N^{1/2} e_m$ after N operations.
  - But algorithms rarely random $\rightarrow$ linear error $N e_m$.

Round-off Error, Cont'd

- Subtraction of two very nearly equal numbers can give rise to large RE.

  e.g. Solution of quadratic equation...

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  ...can go badly wrong whenever $ac \ll b^2$ (Cf. PS#2).

- RE cannot be avoided—it is a consequence of using a finite number of bits to represent real values.

Truncation Error

- In practice, most numerical algorithms approxi-mate desired solution with a finite number of artithmetic operations.

  e.g. evaluating integral by quadrature

  summing series using finite number of terms

- Difference between true solution and numerical approximation to solution is called "truncation error" (TE).

Truncation Error, Cont'd

- TE exists even on "perfect" machine with no RE.
- TE is under programmer's control; much effort goes into reducing it.
- Usually RE and TE do not interact.
- Sometimes TE can amplify RE until it swamps calculation. Solution is then called <u>unstable</u>.

  e.g. Integer powers of Golden Mean (Cf. PS#2).