Random Numbers

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- *NRiC* §7.
- Frequently needed to generate initial conditions.
- Often used to solve problems statistically.
- How can a computer generate a random number?
  - It can't! Generators are *pseudo-random*.
  - Generators are *deterministic*: it's always possible to produce the same sequence over and over.
  - Sometimes this is a good thing!

Random Number Generators

- User specifies an initial value, or *seed*.
- Initializing generator with same seed gives same sequence of “random” numbers.
- For a different sequence, use a different seed.
- One strategy is to use the current time, or the processor ID, to seed the generator.
  - Problem: this may have poor dynamic range, or may be correlated with when the code is run.
  - Solution: *combine* sources, e.g., `int seed = (int) time(NULL) % getpid() + getppid()`, to get a more robust seed.

Choosing a Generator

- Since generators do not produce truly random sequences, it's possible that your results may be affected by the generator used!
- Often the supplied generators on a given machine have poor statistical properties.
- But even a statistically sound generator can still be inadequate for a particular application.
- Be wary if you ever need more than $\sim 10^6$ random numbers, and certainly if you need more than the largest representable integer!
- Solution: always compare results using two generators.

Guidelines

- Follow these steps to minimize problems:
  1. Always remember to seed the generator before using it (discarding any returned value).
  2. Use seeds that are “somewhat random,” i.e., have a good mixture of bits, e.g., 2731771 or 10293085 instead of 1 or 4096 or some other power of 2.
  3. Avoid sequential seeds: they may cause correlations.
  4. Compare results using at least two generators.
  5. When publishing, indicate generator used.
  6. Often it’s a good idea to make a note of the seed used for a given run, in case you need to regenerate the sequence again later.

Uniform Deviates

- Random numbers that lie within a specified range (typically 0 to 1), with any one number in the range as likely as any other, are *uniform deviates*, i.e.,

$$p(x)\,dx = \begin{cases} dx & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

- Useful in themselves, often used to generate differently distributed deviates.
- Distinguish between linear generators (discussed next) and nonlinear generators (do a web search).

Linear Congruential Generators

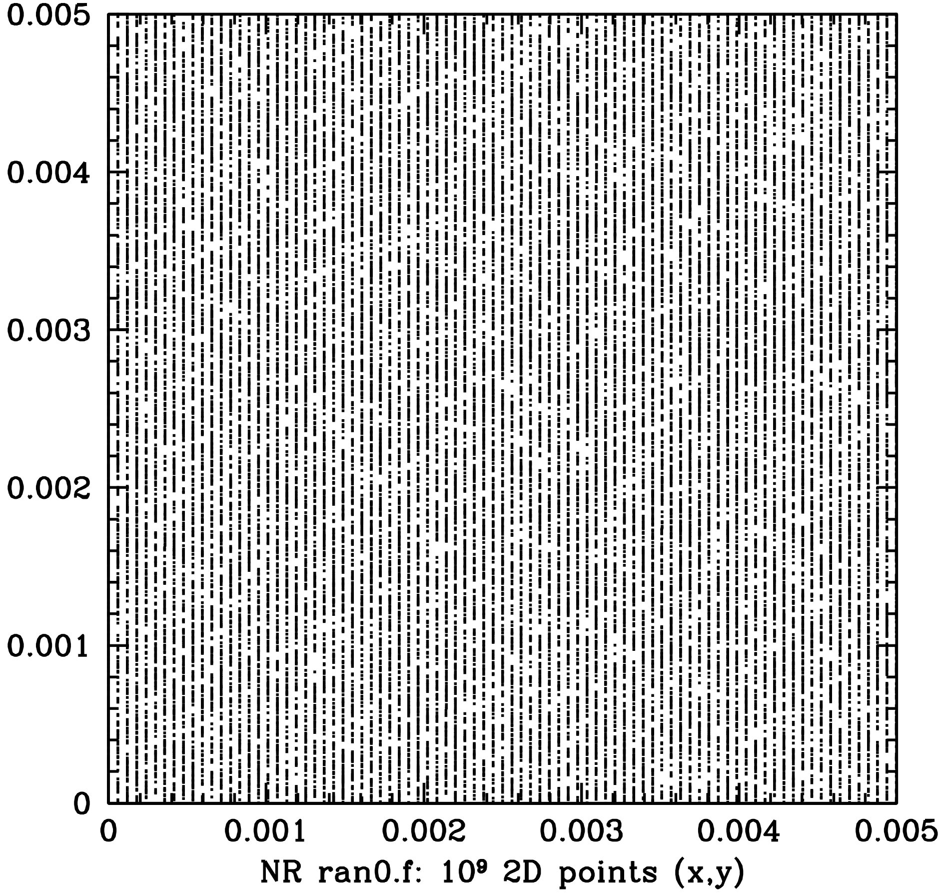
- Typical of most system-supplied generators.
- Produces series of integers I_1, I_2, I_3, ..., each between 0 and $m - 1$, using:

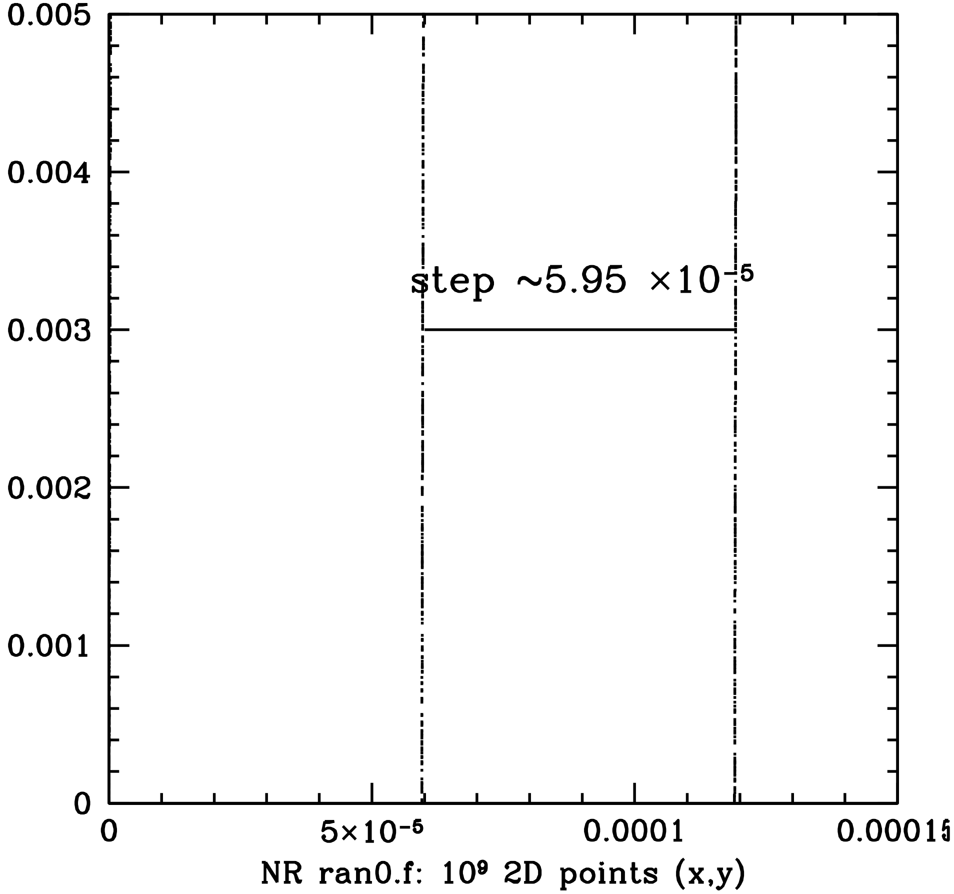
$$I_{j+1} = aI_j + c \pmod{m},$$

  where m is the modulus, and a and c are positive integers called the multiplier and the increment, respectively.
- If m, a, and c are properly chosen, all possible integers between 0 and $m - 1$ occur at some point.
  - The choice of $a = 7^5 = 16807$, $c = 0$, $m = 2^{31} - 1 = 2147483647$ is known as the minimal standard generator.
  - Often a and c chosen so as to have integer overflow on nearly every step, giving less predictable sequence and avoiding the mod operation.

- The LCG method is very fast but it suffers from sequential correlations.
- If k random numbers at a time are used to plot points in k-dimensional space, points tend to lie on $(k-1)$-dimensional hyperplanes. There will be <u>at most</u> $m^{1/k}$ planes, e.g., $\sim$ 1600 if $k = 3$ and $m = 2^{32}$!
- The quality of a LCG is measured by the maximum distance between successive hyperplanes: the smaller the distance, the better.

Example: ran0.f





- Also, low-order bits may be less random than high-order bits, e.g., last bit alternating between 0 and 1.
  - To generate random number between 1 and 10 with `rand()`, use
    ```
    j = 1 + (int) (10.0*rand()/(RAND_MAX + 1.0));
    ```
    and *not*
    ```
    j = 1 + (1000.0*rand() % 10);
    ```
    (which uses lower-order bits).

NRiC RNGs

- *NRiC* gives several uniform deviate generators:

Generator	Speed	Notes
`ran0`	1.0	Small multiple, serial correlations.
`ran1`	1.3	General purpose, maximum 10^8 values.
`ran2`	2.0	Like `ran1`, but longer period.
`ran3`	0.6	Subtractive method, not well studied.
`ranqd1`	0.1	Fast, machine-dependent.
`ranqd2`	0.3	Ditto.
`ran4`	4.0	Good properties, slow.

- On the department machines, see `rand()`, `random()`, and `drand48()`.
- There is much discussion on the web of relative merits of RNGs. Recommended generators include TT800 and the Mersenne Twister.
- Bottom line: test it yourself, or use web-published testing routines, e.g., spectral methods.

Transformation Method

- Suppose we want to generate a deviate from a distribution $p(y)\,dy$, where $p(y) = f(y)$ for some positive and normalized function f, with y ranging from $y_{\min}$ to $y_{\max}$.
- Let $F(y)$ be the *cumulative* distribution of $f(y)$, from $y_{\min}$ to y, i.e., $F(y) = \int_{y_{\min}}^{y} f(y')\,dy'$.
- Set a uniform deviate $x = F(y)/F(y_{\max})$ and solve for y: this is the new generation function.
- Only useful if $F^{-1}(x)$ is easy to compute.

Example: Exponential deviates

- Suppose we want $p(y)\,dy = e^{-y}\,dy$, $y \in [0, \infty)$.
- Apply the transformation method:
  - Have $f(y) = e^{-y}$, $F(y) = e^{-0} - e^{-y} = 1 - e^{-y}$.
  - Set $x = F(y)/F(\infty)$ and solve $x(1 - e^{-\infty}) = 1 - e^{-y}$ for y.
  - Get $y(x) = -\ln(1 - x) = -\ln(x)$ (since $1 - x$ is distributed the same as x).
- So if x is a uniform deviate between 0 and 1, $y(x)$ ($x > 0$) will be an exponential deviate.
- See *NRiC* §7.2 for Gaussian deviates.

Another example: A simple IMF

- Suppose we want to generate particle masses according to $M\,dM = M^\alpha\,dM$, $M \in [M_{\rm min}, M_{\rm max}]$.
- From the transformation method we get:

$$M = M_{\rm min}\left\{1 + x\left[\left(\frac{M_{\rm max}}{M_{\rm min}}\right)^{\alpha+1} - 1\right]\right\}^{\frac{1}{\alpha+1}},$$

or

$$M = \left[(1-x)M_{\rm min}^{\alpha+1} + xM_{\rm max}^{\alpha+1}\right]^{\frac{1}{\alpha+1}}.$$

- Notice that for a flat distribution ($\alpha = 0$), get expected result.
- What happens if $\alpha = -1$? EFTS...

Initial Conditions

- Often want to generate random initial conditions for a simulation, e.g., initial position and velocity.
- Must take care when using transformations, since may not get distribution you expect.
- For example, to fill a flat disk of radius R with random points is it better to:
  1. Choose random θ and r then set $x = r\cos\theta$, $y = r\sin\theta$?
  2. Fill a square and reject points with $x^2 + y^2 > R^2$?

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  Answer: 2, but 1 will work if r^2 (instead of r) has a uniform random distribution.

Application: Cryptography

- A simple encryption/decryption algorithm can be constructed using random number generators.
- If both parties know the initial seed, they can both reproduce the same sequence of values.
- However, communicating the *seed* between parties carries risk.
- One popular technique is to combine *public* and *private* keys for secure communication (the example below is called Diffie-Hellman Key Exchange).
- How do public and private keys work?

Step	You	Your Friend
1	Public: choose large prime p.	Public: choose b, no common factors with $p-1$.
2	Private: choose x.	Private: choose y.
3	Compute $b^x \bmod p$ and send.	Compute $b^y \bmod p$ and send.
4	Compute $k = b^{yx} \bmod p$.	Compute $k = b^{xy} \bmod p$.

- k is the encryption key. This procedure relies on the fact that is is very difficult to factor large numbers.
- Also uses the handy relationship:

$$(b^y \bmod p)^x \bmod p = (b^y)^x \bmod p, \text{ for any } x, y.$$