N-body Techniques

Part 4

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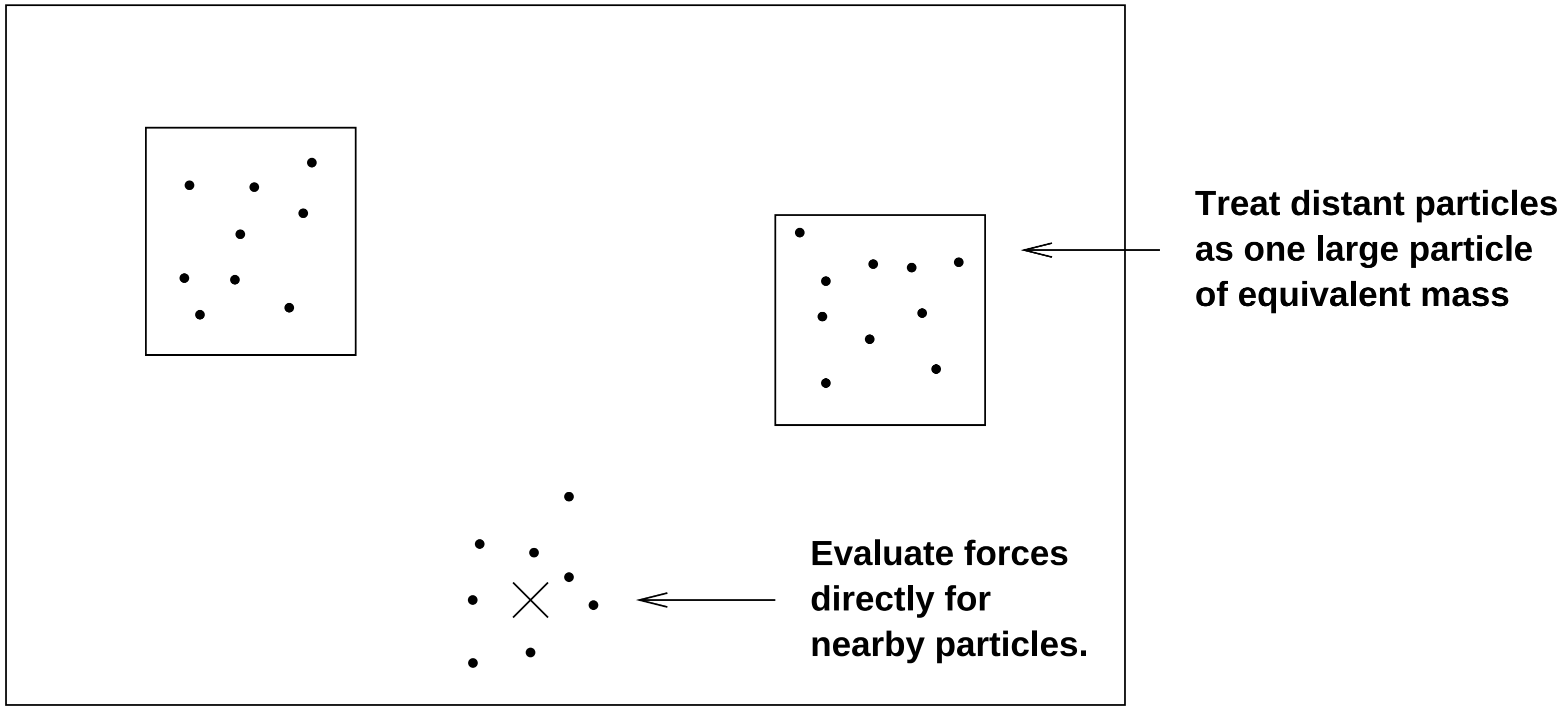
Tree Codes

Efficiency can be increased by grouping particles together:

Nearest particles exert greatest forces $\rightarrow$ direct summation.

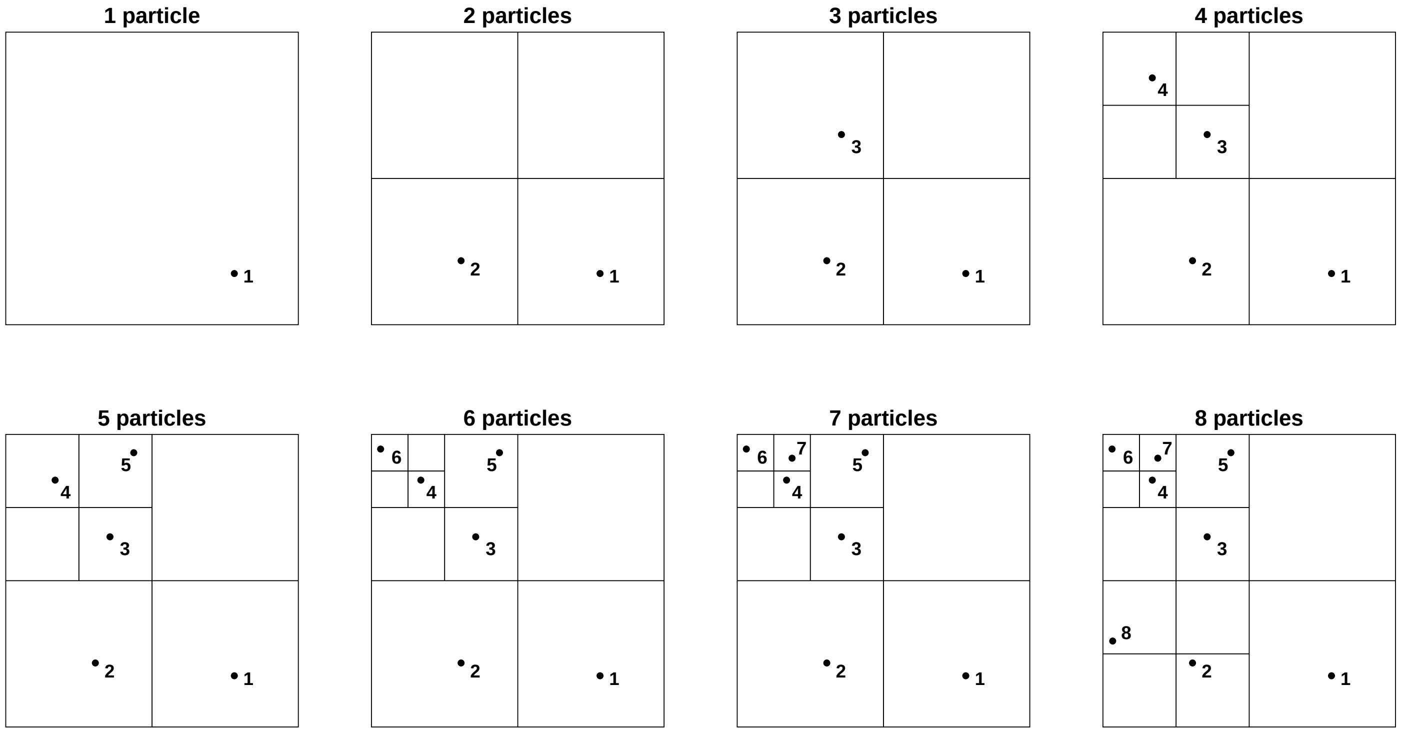
Distant particles exert smallest forces $\rightarrow$ treat in groups.


Treat distant particles as one large particle of equivalent mass
Evaluate forces directly for nearby particles.

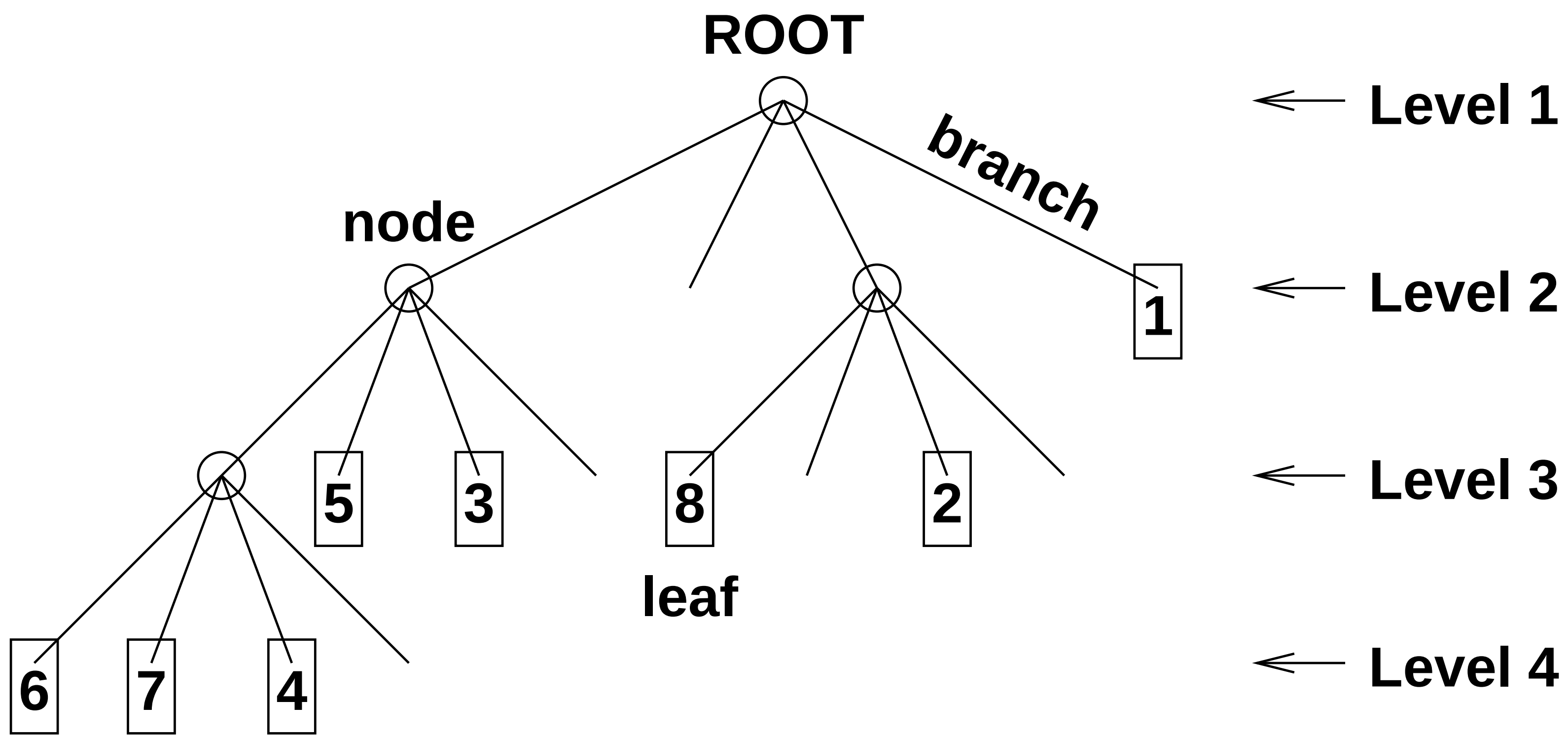
But how do we organize particles into groups? Will sketch one method (Barnes & Hut 1986, *Nature* **324**, 426; also see Hernquist 1987, *ApJS* **64**, 715), then go into more detail.

Barnes & Hut method: Overview

- The BH method is a hierarchical force-calculation algorithm:
  - Place particles on mesh one at a time.
  - Divide mesh into equal volume subdomains at each placement so that each particle occupies a single subdomain. E.g., in 2-D:



- Now, organize particles based on nesting of subdomains:



- How does this speed up force evaluation? Consider evaluation of force on particle 1:
    - If any subdomain subtends an angle $\theta = l/d \lesssim \theta_{\mathrm{crit}}$ as seen from particle 1 (l is size of subdomain, d is distance from particle 1), then treat all particles in that subdomain as one. E.g.,

      Particle 2, 8: treat directly.

      Top-left subdomain: treat as group.

      $\Longrightarrow$ just 3 summations, instead of 7.

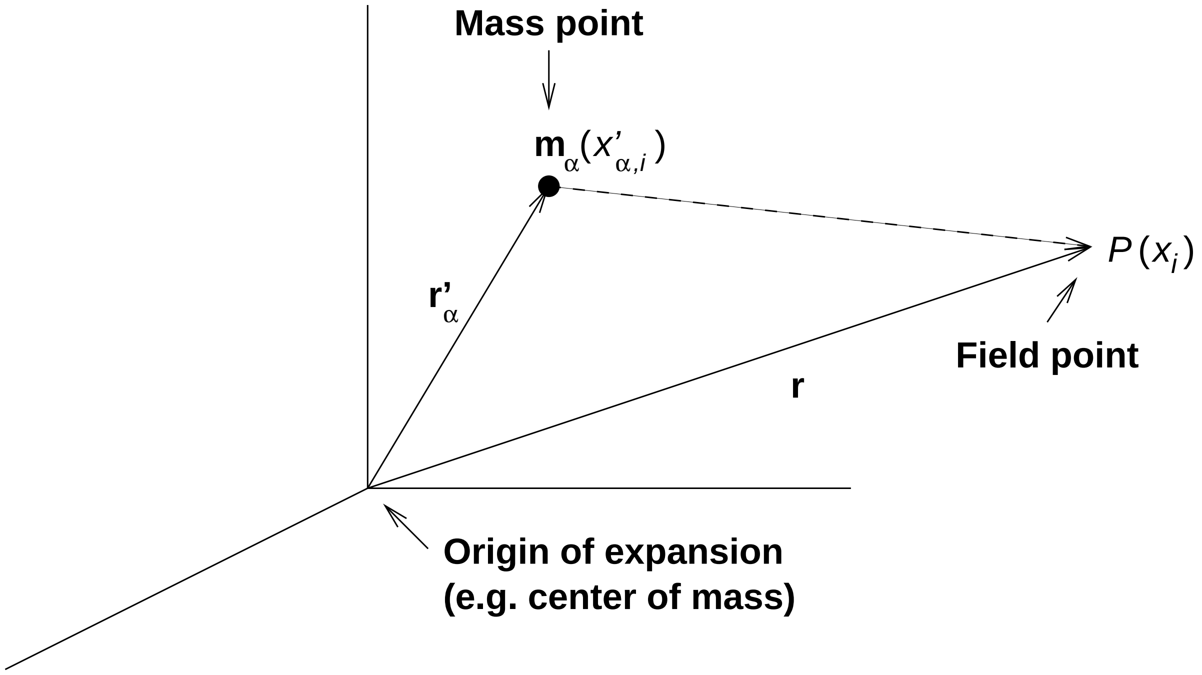
Barnes & Hut method: Details

- Average size of a particle-bearing cell is of order the interparticle spacing $d \sim s/N^{1/k}$ (in k-D) and number of cells in any dimension $\sim s/d$, so number of levels $\sim \mathcal{O}(\log_{2^k} N^{1/k}) = \mathcal{O}(\log N)$.
- $\therefore$ time required to construct tree $\sim \mathcal{O}(N \log N)$.
- Must also compute total mass and center-of-mass position $\Longrightarrow$ one more $\mathcal{O}(N \log N)$ pass through tree.
- Finally, force evaluation ("pruning") $\Longrightarrow$ $\mathcal{O}(\log N)$ sums per particle $\Longrightarrow \mathcal{O}(N \log N)$ scaling $\ll N^2$ for $N \gg 1$.

How bad an approximation is it?

- Consider expanding potential of cell α (e.g., Marion & Heald 1980, pp. 38–40):

$$\Phi_\alpha = -\frac{Gm_\alpha}{r} + Gm_\alpha \sum_i x'_{\alpha,i} \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) - \frac{Gm_\alpha}{2} \sum_{i,j} x'_{\alpha,i} x'_{\alpha,j} \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) + \cdots$$



so

$$\Phi = \sum_\alpha \Phi_\alpha = \Phi^{(1)} + \Phi^{(2)} + \Phi^{(4)} + \cdots + \Phi^{(2^l)} + \cdots$$

where

$$\Phi^{(1)} \equiv -\sum_\alpha \frac{Gm_\alpha}{r} = -\frac{GM}{r} \text{ is the “monopole”,}$$

$$\Phi^{(2)} \equiv \sum_\alpha Gm_\alpha \sum_i x'_{\alpha,i} \frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) \text{ is the “dipole”,}$$

$$\Phi^{(4)} \equiv -\frac{1}{2}\sum_\alpha Gm_\alpha \sum_{i,j} x'_{\alpha,i} x'_{\alpha,j} \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right) \text{ is the “quadrupole”,}$$

$$\Phi^{(2^l)} \equiv \frac{(-1)^{(l+1)}}{l!}\sum_\alpha Gm_\alpha \sum_{i,j,\ldots,l} x'_{\alpha,i} x'_{\alpha,j} \cdots x'_{\alpha,l} \frac{\partial^l}{\partial x_i \partial x_j \cdots \partial x_l}\left(\frac{1}{r}\right)$$

is the “2^l-pole”.

- If we choose expansion center to be center of mass of group, then $\sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0$. But then notice that $\Phi^{(2)} = \sum_\alpha G m_\alpha \mathbf{r}'_\alpha \cdot \nabla(1/r) = 0$, so dipole vanishes. $\therefore$ error term dominated by quadrupole.
- (Can also write
$$\Phi = -\frac{GM}{r} - \frac{1}{2}\frac{G}{r^5}(\mathbf{rQr}),$$
where
$$Q_{ij} = \sum_k m_k(3x_{k,i}x_{x,j} - r_k^2\delta_{ij})$$
is the traceless quadrupole tensor, k is over the mass components, and $\mathbf{r}_k$ is relative to the cell center of mass. With this notation, and invoking the parallel axis theorem, the quadrupole of a parent cell can be constructed via the quadrupoles of its daughter cells: $\mathbf{Q} = \sum_i \mathbf{Q}_i + \sum_i m_i(3\mathbf{r}_i\mathbf{r}_i - r_i^2\mathbf{1})$, where i is over the daughter cells and $\mathbf{r}_i$ is relative to the parent center of mass.)
- Often, quadrupole not needed (monopole is “good enough”).

- With quadrupole, for $\theta_{\rm crit} = 1$, forces typically accurate to $\sim$ 1% (in practice, keep $\theta_{\rm crit} < 1/\sqrt{2} = 0.7$ for 2-D tree, $< 1/\sqrt{3} = 0.6$ for 3-D tree). This is *average* error; certain pathological configurations can give much larger errors. Also, trees in general break $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$...
- For high precision, might consider octopole.
  - Turns out the octopole does *not* help convergence much—need to go to next higher order, the hexadecapole!
  - Obviously this means many more computations to compute force (still scales as $\mathcal{O}(N \log N)$), but can use larger $\theta_{\rm crit}$.
- On balance, probably *never* need better than hexadecapole.

Barnes & Hut method: Pseudocode

Define a node `struct`: contains size, center, mass, position, $\mathbf{Q}$, etc. of cell, plus info on children (may be nodes).

<u>Tree build</u> — start with special cell ("root")

```
start
   root = new node [includes initialization]
   loop over particles i
      put_in_tree(i,root)
   calc_moments(root)

function put_in_tree(particle,node)
   which octant (child) contains particle?
   is child...
      ...empty?  : make particle a leaf
                   break
```

```
    ...leaf?   : make leaf a branch
                 child = new node
                 put_in_tree(leaf,child)
    ...branch? : put_in_tree(particle,child)

function calc_moments(node)
  [loop over non-empty child cells
   is child...
    ...leaf?   : node->mass += leaf->mass
                 node->pos += (leaf->mass)*(leaf->pos)
                 break
    ...branch? : calc_moments(child)
                 node->mass += child->mass
                 node->pos += (child->mass)*(child->pos)
  ]
   node->pos /= node->mass
```

<u>Tree walk</u> — start at root

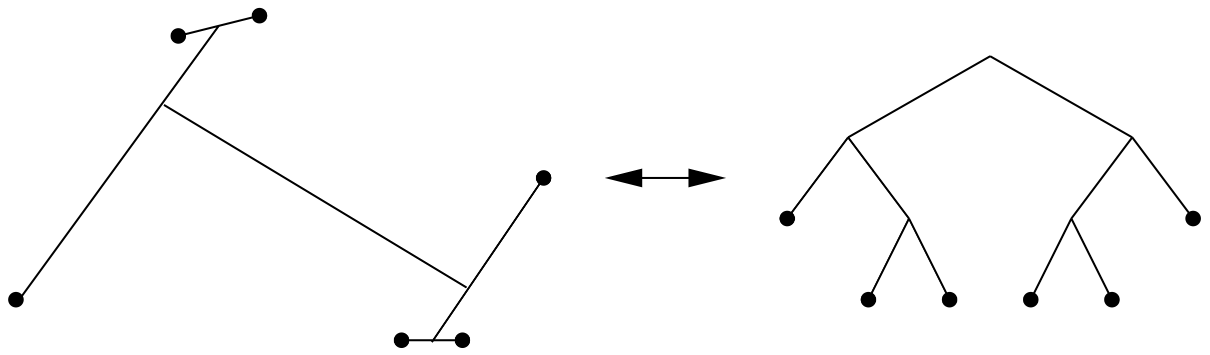
```
function add_force(pos,node,force)

compute theta = (node->size)/(distance to node)

theta < theta_crit? : force += expansion(node) ["prune"]
              else  : [loop over non-empty child cells
                       is child ...
          ... leaf?   : force += (direct force)
                        break
          ... branch? : add_force(particle,child,force)
                       ]
```

Other Types of Trees

- Differ primarily in organization of particle information.

Mutually nearest neighbour

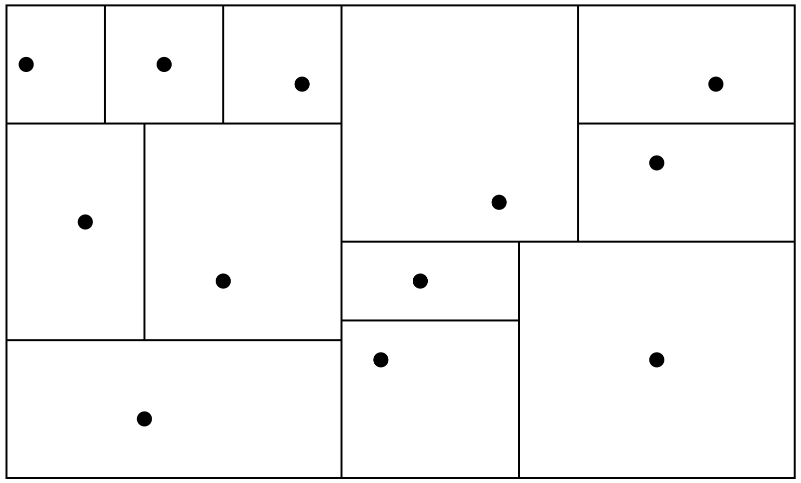
- E.g., Appel 1981, Jernigan 1985, Porter 1985.
- Given N particles, two nearest joined together $\rightarrow$ node, leaving $N-1$ entities ($N-2$ particles plus 1 node) in list.
- Node contains total mass and center-of-mass position of cluster.
- Repeat until only 1 cluster remains.
- $\mathcal{O}(\log_2 N)$ levels (binary tree), $\mathcal{O}(N \log N)$ update time.



- Advantage: Preserves physical proximity of particles (binaries). Can also let particles “drift” a while before update.
- Disadvantage: Arbitrary node shapes, hard to estimate error.

k-D tree (recusively bisect longest dimension)

- E.g., Olson & Packer 1996.
- First determine dimension (x, y, or z) that spans largest spatial range of particle distribution.
- Sort data on this dimension and divide into halves containing equal numbers of particles.
- Repeat with sublists until each contains only 1 particle.
- Often used for “domain decomposition” to balance work between multiple processors.



- Advantage: No empty cells, more efficient shape.
- Disadvantage: Extreme oblong shapes $\rightarrow$ larger error.

Summary

- PP method (direct summation) — most accurate, but $\mathcal{O}(N^2)$.
- PM method — $\mathcal{O}(N_g \log N_g)$, but resolution limited.
- Tree codes — $\mathcal{O}(N \log N)$, but sometimes difficult to implement.
- Also: PP-PM = P^{3}M — direct summation over nearby particles, use grid for distant interactions.