ASTR615 Fall 2015 — Problem Set #5

Due Wed November 18th, 2015

Write a program to integrate any number of coupled differential equations using the Euler method, fourth-order Runge-Kutta, and Leapfrog (note: Leapfrog only applies to special cases). You will be using this program in a future assignment, so make sure it's well documented. It's recommended that you use double precision throughout.

1. Use your program to solve the following differential equation for $x(t)$:

$$\frac{d^2x}{dt^2} + x = 0,$$

with initial conditions $x(0) = 0$, $\dot{x}(0) = 1$. Note the analytical solution is $x = \sin(t)$.

   (a) Integrate the equation for $0 \leq t \leq 15$ using each of the methods, and step sizes of 1, 0.3, 0.1, 0.03, and 0.01.

   (b) Plot your integration results against the analytical solution for each case. (*Hint*: do all the Euler plots on one page, with one plot per timestep; then all the Leapfrog plots on another page, etc.) Comment on the results.

   (c) Plot $\log|x_{\text{numerical}}(15) - x_{\text{exact}}(15)|$ as a function of $\log(\textit{stepsize})$ in each case and comment. (*Hint*: does the error have the expected dependence on the stepsize? Remember you're integrating over many steps, not just one.)

2. Now try the two-dimensional orbit described by the potential:

$$\Phi = -\frac{1}{\sqrt{1 + 2x^2 + 2y^2}},$$

where we are assuming unit mass for the particle in this potential. Show analytically that the orbits are given by the coupled differential equations:

$$\frac{d^2x}{dt^2} = -\frac{2x}{(1 + 2x^2 + 2y^2)^{3/2}}$$

$$\frac{d^2y}{dt^2} = -\frac{2y}{(1 + 2x^2 + 2y^2)^{3/2}}$$

and then reduce these to 4 coupled first-order equations.

   (a) Integrate this system for $0 \leq t \leq 100$ with the initial conditions $x = 1$, $y = 0$, $\dot{x} = 0$, $\dot{y} = 0.1$. Try this with Leapfrog and Runge-Kutta, and step sizes of 1, 0.5, 0.25, and 0.1. Plot x *vs.* y for these integrations.

   (b) Plot the energy $E = (\dot{x}^2 + \dot{y}^2)/2 + \Phi(x, y)$ as a function of time for your integrations.

3. Plot phase diagrams (x *vs.* y) for the Lotka-Volterra Predator-Prey model:

$$\begin{aligned}\dot{x} &= Ax - Bxy - dx \\ \dot{y} &= -Cy + Dxy - ey\end{aligned}$$

where x is the prey density (rabbits), y the predator density (foxes), $A = 1$ (rabbit reproduction rate), $B = 0.1$ (rabbit consumption rate by foxes), $C = 1.5$ (fox death rate by natural causes), $D = 0.03$ (fox population growth rate due to consumption of rabbits), and $d = e = 0$ (hunting rate of foxes and rabbits, respectively). Use only your Runge-Kutta integrator, with $t = 0$ to 100 and timesteps of 1, 0.5, 0.25, and 0.1 to solve this system, starting with $x = 30$ and $y = 3$. If $d = e = q$, for roughly what value of q do both populations drop below 10^{-9} by $t = 100$ for a timestep of 0.1?