(8) Kinetics in expanding universe

ERA OF NORMAL PHYSICS:

Kinetics in the expanding Universe:

After reheating thermal equilibrium is established. Radiation-dominated (relativistic particles) era:

$$a(t) \propto t^{1/2}$$

Boltzmann equation in expanding Universe:

particle x interacts with particle Y (eg, $Y = \bar{x}$: $\bar{x}$ = antiparticle of x)

reaction terms

$$a^{-3}\frac{d(n_x a^3)}{dt} = \frac{dn_x}{dt} + \underbrace{3Hn_x}_{\text{Hubble expansion term}} = \langle v\sigma \rangle n_Y (n_x^{eq} - n_x)$$

more precisely: $1+2 \leftrightarrow 3+4$

$$\langle v\sigma \rangle \cdot n_1^{eq} n_2^{eq} \cdot \left(\frac{n_3}{n_3^{eq}} \cdot \frac{n_4}{n_4^{eq}} - \frac{n_1}{n_1^{eq}} \cdot \frac{n_2}{n_2^{eq}} \right)$$

- Expansion and reaction time scales:

1) $t_H \sim \frac{1}{H} \sim 2t \propto a^2 \quad \left(a \propto t^{\frac{1}{2}}\right)$

2) $\tau = \frac{1}{\langle v\sigma \rangle n_Y}$

- If $\tau \ll t_H \rightarrow$ we can neglect expansion $\boxed{n_X \equiv n_X^{eq}}$

Assume $\langle v\sigma \rangle \propto T^{\alpha} \Rightarrow \tau \propto a^{3+\alpha} \quad (n_Y \propto a^{-3}; T \propto a^{-1})$

$$\Downarrow$$

$\frac{\tau}{t_H} \propto a^{1+\alpha} \ll 1$ for $a \ll a^*$.

and $\underbrace{\alpha > -1}$

reasonable assumption !

$a \to 0$ reactions in equilibrium.

later times $\boxed{t > t_H}$ $\quad \left|\langle \sigma v \rangle n_Y (n_X^{eq} - n_X)\right| \ll H n_X$

$$\Downarrow$$

$$\frac{dn_X}{dt} + 3Hn_X = a^{-3}\frac{d\, n_X a^3}{dt} \simeq 0 \Rightarrow n_X a^3 = \text{const}$$

$\bar{n}_X \equiv n_X a^3 =$ comoving density $=$ const. (freeze-out)

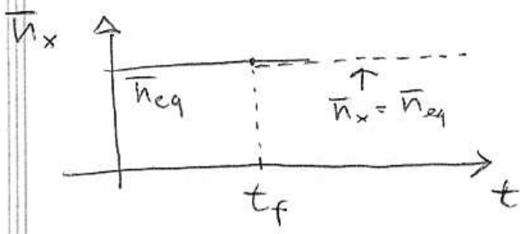
definition: $\frac{\tau(t_f)}{t_H(t_f)} \simeq \tau(t_f) \cdot H(t_f) = 1$

$$\boxed{\bar{n}_{X,\infty} \simeq \bar{n}_X^{eq}(t_f)}$$

Examples:

a) relativistic particles: ($mc^2 \ll k_B T$)

$n_{eq}(T) \sim T^3 \sim a^{-3}$ $\qquad \bar{n}_{eq}(T) = \text{const}$

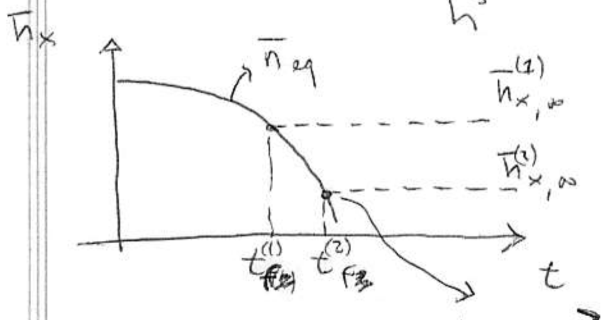


always at the equilibrium abundance if decouple relativistic

b) non-relativistic $\quad mc^2 \gg k_B T$

$$n_{eq}(T) \sim \frac{(2\pi k_B T m)^{3/2}}{h^3} e^{-\frac{mc^2}{k_B T}}$$

(Boltzmann dist.)



$\langle \sigma_2 v_2 \rangle > \langle \sigma_1 v_1 \rangle$

- more strongly interacting $\rightarrow$

$t_f^1 < t_f^2 \rightarrow n_x^{(1)} > n_x^{(2)}$

$t_f^{(1)} < t_f^{(2)}$

$n_x^{(1)} > n_x^{(2)}$

Abundance of strongly interacting particles is lower than weaker interacting part.

SKIP?

Thermal history

0) At $T \gg 100$ GeV Higgs field ← $\boxed{\text{infation}}$ (unknown physics) (*)

After reheating:

1) At $T \simeq 100$-200 MeV standard set of "low energy physics" particles:

$\Rightarrow p, n; e^{\pm}; \nu_e, \bar{\nu}_e; \nu_\mu; \bar{\nu}_\mu; \nu_\tau; \bar{\nu}_\tau; \underbrace{\mu^{\pm}; \pi^{\pm}; \pi^0}_{\text{mesons}}$

Hadron era Π in thermodynamic equilibrium

2) At $T \simeq 130$ MeV $\pi^{\pm} \rightarrow$ annihilate
$\pi^0 \rightleftarrows \gamma$ decay } lepton era

p and n non-relativistic and non-equilibrium.

(*)
1) electromagnetic int: $p+\bar{p} \rightleftarrows n+\bar{n} \leftrightarrow \pi^+ + \pi^- \leftrightarrow \mu^+ + \mu^- \leftrightarrow e^+ + e^- \leftrightarrow \pi^0 \leftrightarrow \gamma$

2) weak interaction: $e^- + \mu^+ \leftrightarrow \nu_e + \nu_\mu$ $e^+ + e^- \leftrightarrow \nu_e + \bar{\nu}_e$
$e^- + p \leftrightarrow \nu_e + n$ $e^{\pm} + \nu_e \leftrightarrow e^{\pm} + \nu_e$
$\mu^- + p \leftrightarrow \bar{\nu}_\mu + n$ ......

- Cross sections:

EM → $\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \sim 6.65 \cdot 10^{-25} \left(\frac{m_e}{m}\right)^2 \text{cm}^2$

WEAK → $\sigma_{wk} = g_{wk}^2 \left[\frac{k_B T}{(\hbar c)^2}\right]$ ($g_{wk} \simeq 1.4 \cdot 10^{49}$ erg cm^3)

Neutrino decoupling:

μ mass $\simeq 106$ MeV → $\mu^{\pm}$ annihilate shortly after Π

$\mu^- + \mu^+ \leftrightarrow \nu_\mu + \bar{\nu}_\mu$ $\nu_e + \mu^- \leftrightarrow \bar{\nu}_\mu + e^-$
$\mu^+ \leftrightarrow e^+ + \bar{\nu}_\mu + \nu_e$ $\bar{\nu}_\mu + \mu^+ \leftrightarrow \nu_e + e^+$
$\mu^- \leftrightarrow e^- + \nu_\mu + \bar{\nu}_e$ ......

Muon life time $\tau_\mu \sim 2 \cdot 10^{-6}$ sec

- hadron era (quarks)
  $\gamma, \nu_e, \bar{\nu}_e, \nu_\mu; \bar{\nu}_\mu; \nu_\tau, \bar{\nu}_\tau, e^-, e^+, \mu^-, \mu^+ \pi^-, \pi^0, \pi^+$
- lepton era (neutrinos)

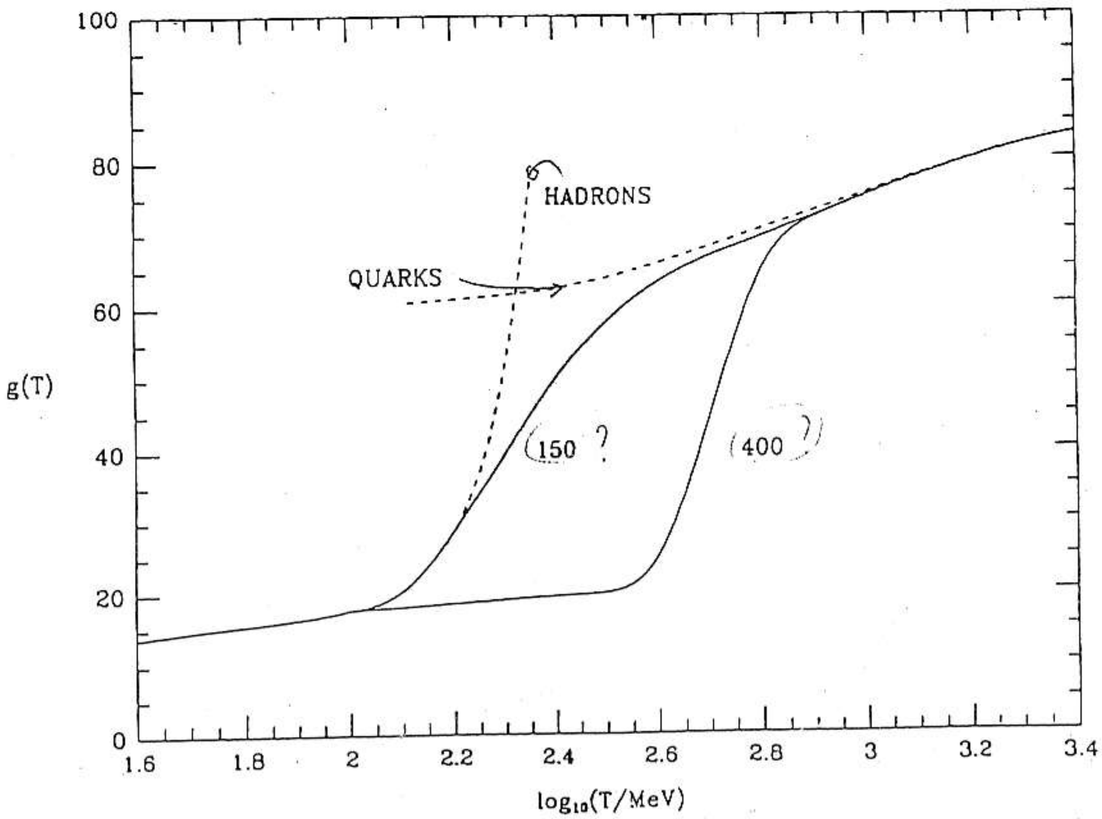
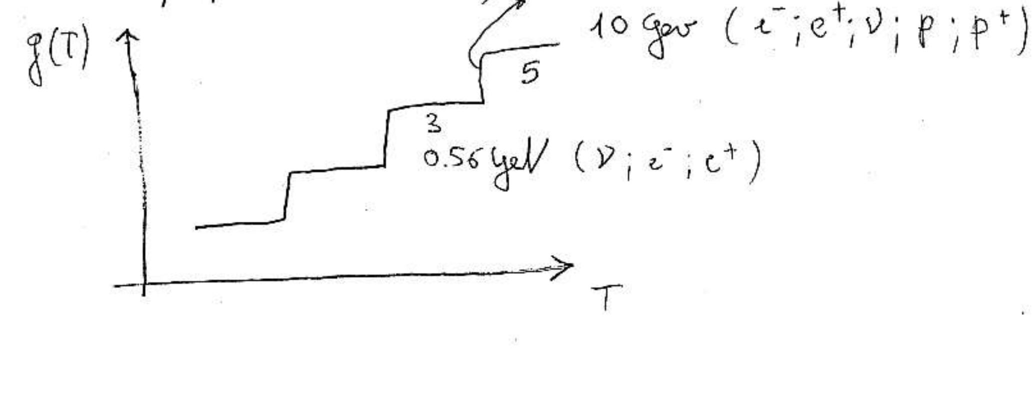


Fig. 2. The effective numbers of relativistic degrees of freedom as a function of temperature. The dashed lines correspond to free quarks and hadrons. (this picture is smoothed ....)

from: Olive, K. A., 1994, in "Matter under extreme conditions", (Springer-Verlag : Berlin)

(also hep-ph/9404352)



(65B)

$e^- + e^+ \to \gamma; \nu$

Temperature range		Particles	$4g(T)$
$T < m_e$	=0.511 MeV	γ, ν	29
$m_e < T < m_\mu$	=106 MeV	e	43
$m_\mu < T < m_\pi$	=135 MeV	μ	57
$m_\pi < T < T_{q\to b}$	=150-400 MeV ?	π	69
$T_{q\to b} < T < m_s$	=100-300 MeV	u, d, gluons (-π)	205
$m_s < T < m_c$	=1.0-1.6 GeV	s	247
$m_c < T < m_\tau$	=1.777 GeV	c	289
$m_\tau < T < m_b$	=4.1-4.5 GeV	τ	303
$m_b < T < m_{W,Z}$	=80 GeV	b	345
$m_{W,Z} < T < m_t$	>62 GeV	W, Z	381
$m_t < T < m_H$	>58 GeV	t	423
$m_H < T$		H^0	427

Higgs field (scalar field)

e	electron, positron
ν	neutrinos (3×2 species)
μ	muon
π	π-mesons ($\pi^{0,\pm}$)
$T_{q\to b}$	temperature of the quark-baryon transition
u	u-quark
d	d-quark
s	strange quark
c	charmed quark
τ	τ-lepton (taon)
b	bottom quark
W	$W^\pm$ bosons
Z	Z-boson
t	top quark
H	Higgs field

Adopted from Olive, K. A. 1994, in Matter Under Extreme Conditions (Berlin: Springer-Verlag); hep-ph/9404352.

(9) Thermal history and ν decoupling

- ~~After reheating ($T > 10^9$ GeV) $\rightarrow$ Lepton era: from pion~~ annihilation to $e^- e^+$ annihilation.

- **Energy density of relativistic matter:**

$$\epsilon_{rad} = \sum \rho_i c^2 = \sum_i f_s \, g_i \, a_R T^4 \qquad a_R = 7.56 \cdot 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$$

- g_j = effective # degrees of freedom (internal)

$$\begin{cases} \gamma \text{ and } e^{\pm} \rightarrow g = 2 & \text{spin and polarisation} \\ \nu \rightarrow g = 1 & \end{cases}$$

- $f_s = \begin{cases} 1 & \text{bosons} \\ \frac{7}{8} & \text{fermions} \end{cases}$

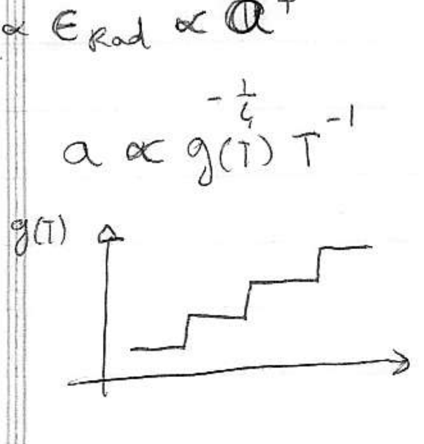
$$\Rightarrow \epsilon_{rad} = \rho_{TOT} c^2 = g(T) \, a_R T^4 \qquad \text{where}$$

$$g(T) = \sum_j^{\text{bosons}} g_j + \frac{7}{8} \sum_j^{\text{fermions}} g_j$$

[$g(T)$ increases at increasing T because at lower T particles – antiparticles annihilate.

$$P_{rad} \propto \epsilon_{Rad} \propto a^{-4} \qquad \text{rad. dominated era } \left(P = \tfrac{1}{3}\rho\right)$$

$$a \propto g(T)^{-\frac{1}{4}} T^{-1} \qquad a \cdot T \propto g(T)^{-\frac{1}{4}} \neq \text{const}$$



$\leftarrow$ see slide

(*) Valid as far as $g(T)$ is constant!

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$$a^{-4} \cdot \epsilon_R(z=0) = (7 \cdot 10^{-13}\, erg/cm^3)\, a^{-4} = \frac{1}{2} a_R\, g(T)\, T^4 \qquad \rho_R \propto a^{-4}$$

$$\Downarrow$$

- $T = \dfrac{3.7\,K}{a \cdot g(T)^{\frac{1}{4}}}$ where $g(z=0) = 1.354$
  $$\left[2 + 2 \cdot \underset{=3}{N_\nu} \frac{7}{8} \left(\frac{4}{11}\right)^{\frac{4}{3}} \right]$$

  $+$

  $t = (2.4 \cdot 10^{19}\, s) \cdot a^2$

  $\Downarrow$

- $t = 2.4\, s \left(\dfrac{1\, MeV}{k_B T \cdot g^{\frac{1}{4}}(T)}\right)^2$ (*)

  (Muon Lifetime $\tau_\mu \sim 2 \cdot 10^{-6}\, s$ at $T > 100\, MeV$)

  thus at $T \simeq 100$ MeV

  $$\boxed{t_{T \sim 100 MeV} \simeq 10^{-4}\, s \gg \tau_\mu}$$

$\rightarrow$ At $T \gtrsim 100$ MeV μ are in equilibrium.

$\rightarrow$ At $T < 100$ MeV $\rightarrow \tau_\mu = 2 \cdot 10^{-6}\, s\; e^{\frac{106\, MeV}{KT}}$

freeze-out at $\tau_\mu \sim t \Rightarrow 2.4 \left(\dfrac{1 MeV}{k_B T\, g^{\frac{1}{4}}}\right) = 2 \cdot 10^{-6}\, e^{\frac{106\, MeV}{KT}}$

since $m_e < T_{\nu_\mu, dec} < m_\mu$

$\rightarrow g(T) \simeq \dfrac{43}{4}$ $\Longrightarrow$ $\boxed{T_{\nu_\mu, dec} \simeq 14\, MeV}$

(see Table in slide)

$0.5\, MeV \simeq m_e < T_{\nu_\mu, dec} < m_\mu \simeq 106\, MeV$

Analogously:

$$\boxed{\begin{array}{l} T_{\nu_e, dec} \simeq 4.2\, MeV \\ T_{\nu_\tau, dec} \simeq 90\, MeV \end{array}}$$

- Important conclusion:

$\Longrightarrow m_e < T_{\mu, \tau, \nu, dec} < m_\mu$ same $g(T)$ $\rightarrow$ $\boxed{T_{\mu, \tau, \nu} = \dfrac{T_{\nu 0}}{a}}$

$$\boxed{T_{\nu_e 0} = T_{\nu_\mu 0} = T_{\nu_\tau 0}}$$

Cosmic Neutrinos background:

a) Before e^+e^- annihilation at $T \simeq 0.5$ MeV

$T_\nu \cdot a = T_{\nu 0} = $ const

b) After $e^+ + e^-$ annihilation:

$T_\gamma \cdot a = T_{\gamma 0} = 2.728$ K $=$ const

Assume entropy is conserved: (reversible process)

$$\underset{\gamma, e^\pm}{S_{\text{MeV}}} = \underset{\gamma}{S_{0.1\,\text{MeV}}}$$

$$S = \underbrace{\left(\frac{\rho c^2 + P}{T}\right)}_{P = \frac{1}{3}\rho c^2} \cdot V = \frac{4}{3}\frac{\rho c^2}{T} \cdot V$$

$$\boxed{\begin{matrix}\rho \propto T^4 \\ S \propto T^3\end{matrix}}$$

$T_\gamma = T_{\nu_0}$

$$\frac{4}{3}\left(\frac{\rho_\gamma}{T} + \frac{\rho_{e^+}}{T} + \frac{\rho_{e^-}}{T}\right) = \frac{4}{3}\frac{\rho_\gamma}{T_{\gamma 0}}$$

$\rho_\gamma \propto T_\gamma^4$, $\rho_{e} \propto T_e^4$

$$\underset{T_{\nu 0}^3}{T_\gamma^3}\left(1 + \frac{7}{4}\right) = T_{\gamma 0}^3$$

at th. equilib.: $\rho_{e^\pm} = \frac{7}{8}\rho_\gamma$ $\quad \left(\epsilon_{\text{ferm}} = \frac{7}{8}\epsilon_{\text{bosons}}\right)$

$$\left(\frac{\rho_{\nu 0}}{T_{\nu 0}}\right) \leftarrow T_{\nu 0}^3\left(\frac{7}{4} + 1\right) = T_{\gamma 0}^3$$

$$\boxed{T_{\nu 0} = \left(\frac{4}{11}\right)^{\frac{1}{3}} \underset{2.728\,\text{K}}{T_{\gamma 0}} = 1.947\ \text{K}}$$

In general:

- Entropy conservation $\Rightarrow S_R a^3 = $ const
- Energy " $\Rightarrow \epsilon_R a^4 = $ const

$$\begin{cases} S = \sum_j \frac{\rho_j c^2 + P_j}{T_j} = \sum \frac{4}{3} f_s g_j a_R T_j^3 = \frac{4}{3} a_R \bar{g} T^3 \\ \epsilon = \sum_j \rho_j c^2 = \sum f_s g_j a_R T_j^4 = a_R g T^4 \end{cases}$$

where:

$$\begin{cases} \bar{g} = \sum_j^{\text{bosons}} g_j \left(\frac{T_j}{T}\right)^3 + \frac{7}{8} \sum_j^{\text{fermions}} g_j \left(\frac{T_j}{T}\right)^3 \\ g = \sum_j^{\text{bosons}} g_j \left(\frac{T_j}{T}\right)^4 + \frac{7}{8} \sum_j^{\text{fermions}} g_j \left(\frac{T_j}{T}\right)^4 \end{cases}$$

At $z=0$:
$$\begin{cases} \bar{g}_0 = 2 + 3 \cdot 2 \cdot \frac{7}{8} \left(\frac{4}{11}\right) = 3.909 \\ g_0 = 2 + 3 \cdot 2 \cdot \frac{7}{8} \left(\frac{4}{11}\right)^{\frac{4}{3}} = 3.363 \end{cases}$$

Thermal History:

$$\boxed{S \cdot a^3 = \text{const}}$$

$(S \cdot V = \text{const})$
↳ specific entropy per unit volume!

$$a^3 \cdot S(T) = S(z=0) \Rightarrow a^3 \bar{g}\, T^3 = \bar{g}(z=0)(2.728\ \text{K})^3$$

$$T = \left(\frac{\bar{g}(z=0)}{\bar{g}(z)}\right)^{\frac{1}{3}} \frac{T_{\gamma 0}}{a} = \left(\frac{3.909}{\bar{g}(z)}\right)^{\frac{1}{3}} \cdot \frac{2.728\ \text{K}}{a}$$

1) $(a = a(T))$

$+$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho \qquad \epsilon_{rad} = \rho c^2 = a_R\, g\, T^4$$

2) $\left(\frac{da}{dt} = f(T)\right)$

$\frac{dT}{dt} = f(T)$ (1)+(2)

$\Downarrow$

$$\boxed{\frac{\dot{T}}{T} = -\left(\frac{8\pi G a_R}{3c^2}\right)^{\frac{1}{2}} \frac{g_{(T)}^{\frac{1}{2}}}{\left(1 + \frac{1}{3} \frac{T}{\bar{g}(T)} \frac{d\bar{g}(T)}{dT}\right)} T^2} \qquad (1)$$

Verify:

$$\left[\begin{array}{l} \ln T = \text{const} - \ln a - \frac{1}{3}\ln \bar{g} \implies \frac{\dot{T}}{T} = -\frac{\dot{a}}{a} - \frac{1}{3}\frac{\dot{\bar{g}}}{\bar{g}} \implies \\ \frac{\dot{T}}{T} = -H - \frac{1}{3}\frac{d\bar{g}}{dT}\frac{dT}{dt}\frac{1}{\bar{g}} \implies \frac{\dot{T}}{T}\left(1 + \frac{1}{3}\frac{T}{\bar{g}}\frac{d\bar{g}}{dT}\right) = -H \end{array}\right.$$

c.v.d

$$\left(H \propto g^{\frac{1}{2}} T^2\right)$$

Integrate (1):

$$\int_{\infty}^{T} \frac{dT'}{T'^3}\left(\frac{1 + \frac{1}{3}\cdots}{g^{\frac{1}{2}}}\right) = -\int_0^t \text{const}\, dt \implies \boxed{\frac{t}{t_*} = \frac{[1+\Delta(T)]}{g^{\frac{1}{2}}(T)}\left(\frac{1\text{K}}{T}\right)^2} \qquad (2)$$

where $t_* = 3.7598 \cdot 10^{20}$ s

$$\Delta(T) = 2T^2 g^{\frac{1}{2}}\left\{\int_T^{\infty} \frac{dT'}{T'^3}\left[1 - \left(\frac{g(T)}{g(T')}\right)^{\frac{1}{2}}\right] + \frac{1}{3}\int_T^{\infty}\frac{dT'}{T'^3}\,\frac{\frac{d\bar{g}}{dT}(T')}{g^{\frac{1}{2}}(T')\,\bar{g}(T')}\right\}$$

Note: $\Delta(T) > 0$

Inverting (2) $\implies T = 1\text{K}\, g(T)^{-\frac{1}{4}}\left(\frac{t_*}{t}\right)^{\frac{1}{2}}[1+\Delta(T)]^{\frac{1}{2}}$

solve interactively $T_0 \simeq 1\text{K}\left(\frac{t_*}{t}\right)^{\frac{1}{2}}$ ....

$T_1 = \ldots\ldots$ etc

(10) Dark matter relics

Non baryonic matter : Dark matter

- Radiation ~~Light~~ is emitted only from ~~baryons~~ p; n; (baryons) and e^-
  $\downarrow$ lepton

- $\Omega_b \simeq 0.02 h^{-2}$ from obs. and nucleosynthesis

Dark matter existence : 1) galaxies rotation curves ~~rotation ...~~ ; 2) $\Omega_0 = \Omega_M + \Omega_\Lambda = 1$

$\Omega_M \simeq 0.3$ $\quad \Omega_\Lambda \simeq 0.7$ $\quad$ 3) Cluster of galaxies CMB; SNe; etc

$\downarrow$

$(\Omega_b << \Omega_m)$ (from nucleosynthesis $\Omega_b \simeq 0.02 h^{-2}$)

- Dark matter < neutrinos; relic X particles that are stable: axion; photinos; gravitinos...

We have already seen:

Massless neutrinos contribute $\Omega_R \sim 10^{-5}$ (negligible)

a) "Hot" dark matter: particles that decouple when relativistic ($m_x c^2 < kT_{dec}$)

~~$m_\nu c^2 < k_B T_{dec}$~~

Let's suppose $m_\nu \neq 0$ but $m_\nu c^2 < k_B T_{\nu, dec} \simeq$ ~~3~~ 1-10 MeV

$$n_{\nu,dec} \frac{3}{4} \frac{8\pi \zeta(3)}{h^3 c^3} T_{0,\nu}^3 = 56.14 \text{ cm}^{-3}$$

$$\rho_{\nu,dec} \simeq \underset{\nu+\bar{\nu}}{2} \cdot n_{\nu,dec} \cdot m_\nu = 2.002 \cdot 10^{-31} \text{ g cm}^{-3} \left(\frac{m_\nu c^2}{1\,eV}\right)$$

or $\boxed{\Omega_\nu = 0.0106\, h^{-2} \left(\frac{m_\nu c^2}{1\,eV}\right)}$

eg. $m_\nu = 4.7\, eV, h = 0.5 \Rightarrow \Omega_\nu \simeq 0.2$

b) "Cold" dark matter: decouple when non-relativistic

$$\frac{dn_x}{dt} + 3H n_x = \langle v\sigma\rangle (n_x^{eq} - n_x)\cdot n_y$$

if $n_y = \bar{n}_x$ antiparticle $\Rightarrow \bar{n}_x = n_x^{eq} + n_x$

$$\frac{dn_x}{dt} + 3H n_x = \langle v\sigma\rangle \left[(n_x^{eq})^2 - n_x^2 \right]$$ ← don't enters in the calculation

$$n_x^{eq} = g_x \left(\frac{2\pi m_x k_B T}{h^2} \right)^{3/2} \cdot e^{-\frac{m_x c^2}{k_B T}} \qquad (0)$$

h ← planck const

$\cancel{\bar{n}_x = n_x \cdot a^3}$ ← ~~comoving density~~

$\cancel{\bar{n}_x^{eq} = n_x^{eq} a^3}$

$\cancel{\frac{d\bar{n}_x}{dt} = \langle v\sigma\rangle}$

Simplified treatment:

1) $\langle \sigma v\rangle (T) = c\sigma_0 \left(\frac{k_B T}{m_x c^2}\right)^{\alpha} = c\sigma_0 X^{-\alpha}$ $\qquad X = \frac{m_x c^2}{k_B T}$

2) $\langle \sigma v\rangle \cdot n_x^{eq} = \frac{1}{\tau_x} \xrightarrow{(0)+(1)} \tau_x = (c\sigma_0)^{-1} X^{\alpha} e^{X} \cdot g_x^{-1} \left(\frac{2\pi m_x^2 c^4}{h^2 c^2} \right)^{-3/2} \cdot X^{3/2}$

3) $\tau_x(t_f) \cdot H(t_f) = 1 \Rightarrow \boxed{\tau_x(t_f) = 2t_f}$ $(\tau = t_H)$ $\qquad H(t) = \frac{1}{2\cdot t}$

4) $t = t_* \frac{(1+\Delta)}{g^{\frac{1}{2}}} \left(\frac{1K}{T} \right)^2 = 3.26\cdot 10^{20} s \, \frac{(1+\Delta)}{g^{\frac{1}{2}}} X^2 \left(\frac{k_B 1K}{m_x c^2} \right)^2$

⇓

(2)+(3)+(4) $\qquad 2t_f = \tau_x(t_f) \Rightarrow e^{X_f} X_f^{\alpha - \frac{1}{2}} = F$

$$F = 12.0 \, \frac{g_x (1+\Delta_f)}{g_f^{\frac{1}{2}}} \left(\frac{m_x c^2}{1\,GeV} \right) \underbrace{\left(\frac{\sigma_0}{10^{-44}\, cm^2} \right)}_{\left(\frac{1.24\cdot 10^9 GeV}{\mu c^2} \right)}$$

where $\sigma_0 \equiv \frac{h^3}{c^2 \mu^2}$

Solve for x_f interactively: natural log

$x_f^{(0)} = 1 \rightarrow x_f^{(1)} = \log(F) - (\alpha - \frac{1}{2}) \log x_f^{(0)} = \log F \rightarrow$

$x_f^{(2)} = \log(F) - (\alpha - \frac{1}{2}) \log x_f^{(1)} = \log F - (\alpha - \frac{1}{2}) \log(\log F)$

... etc

Now calculate $\bar{n}_{x,\infty} = a^3(t_f) \cdot n_x^{eq}(t_f) = a_f^3 \cdot g_x \left(\frac{m_x c^2 \sqrt{2\pi}}{hc}\right)^3 \underbrace{x_f^{-\frac{3}{2}} e^{-x_f}}_{\frac{x_f^{\alpha - \frac{1}{2}}}{F}}$

$\left(a_f = \frac{4.3\,K}{T_f \cdot \bar{g}_f^{\frac{1}{3}}} \right) \Rightarrow a_f^3 \propto \frac{1}{T_f^3 \bar{g}_f}$

$\Rightarrow \bar{n}_{x,\infty} \propto \frac{g_x}{g_f} x_f^3 x_f^{-\frac{3}{2}} \frac{x_f^{\alpha - \frac{1}{2}}}{F}$

$\Rightarrow \boxed{\bar{n}_{x,\infty} = 420\ cm^{-3} \frac{g_x}{\bar{g}_f} \frac{x_f^{\alpha+1}}{F}} + x_f = x_f^{(2)} = \ldots$

$n_\infty(t_0) = \bar{n}_{x,\infty} \Rightarrow \Omega_x h^2 = \underset{x + \bar{x}}{2} \cdot m_x n_\infty \frac{1}{\rho_{crit}}$

Substituting expression for $F \Rightarrow$

$$\boxed{\Omega_x h^2 = 4.3 \frac{g_F^{\frac{1}{2}} x_f^{\alpha+1}}{\bar{g}_f (1 + \Delta_f)} \left(\frac{\mu c^2}{10^6\,GeV}\right)^2}$$

note m_x cancels out

cross section of interaction (annihilation)

Examples:

a) Relic baryons in a baryon-symmetric Universe: End of Hadron era $T \simeq 130\ \mathrm{MeV} = m_\pi$

⇒ If baryogenesis never happened we would still get some n and $\bar{n}$ p and $\bar{p}$ today:

⇒ Cross section for baryons annihilation $\mu \sim m_\pi \sim 135\ \mathrm{MeV}$

$$(g_x = 2 ;\ \Delta_f = 0)$$

$$F \simeq 12 \cdot \frac{2}{\sqrt{15}} \cdot 1 \left(\frac{1.2 \cdot 10^{9}\ \mathrm{GeV}}{1.35 \cdot 10^{-1}\ \mathrm{GeV}} \right)^2 \sim 5 \cdot 10^{20}$$

$$g_f = 15 \qquad m_x \sim 1\ \mathrm{GeV}$$

$$x_f^{(2)} \sim 48 + \frac{1}{2} \ln(48) \simeq 50 \qquad (\alpha = 0)$$

$$\Omega_x h^2 \sim \underbrace{x_f}_{50} \cdot \frac{\sqrt{g_f}}{\bar{g}_f} 1.8 \cdot 10^{-14} \sim 10^{-14} \ll \Omega_R h^2$$

$\Omega_b h^2 \sim 2 \cdot 10^{-2} \gg 10^{-14}$ we need baryogenesis.

b) Cold dark matter particles:

Let's pick a range of masses: $5\ \mathrm{GeV} < m_x c^2 < 80\ \mathrm{GeV}$

$$\Downarrow$$

$$\left[\begin{array}{l} \bar{g}_f \simeq g_f \simeq 300 \\ \underbrace{\Delta_f \simeq 0 ;\ \alpha = 0 \quad g_x = 2}_{\text{assume for simplicity}} \end{array} \right.$$

$$F \simeq 1.4 \left(\frac{m_x c^2}{1\ \mathrm{GeV}} \right) \left(\frac{\sigma_0}{10^{-44}\ \mathrm{cm}^2} \right) \longleftarrow \left(\frac{1.24 \cdot 10^{9}\ \mathrm{GeV}}{\mu c^2} \right)^2$$

We want $\Omega_x \simeq 1$ $h \simeq 0.5$

$$(4.3 \cdot 4) \to h^2$$

$$\Rightarrow 1 \simeq \underbrace{17.2 \cdot \frac{1}{\sqrt{300}}}_{= 1} x_F \left(\frac{Mc^2}{10^6\, GeV}\right)^2$$

$$1.53 \cdot 10^6 \left(\frac{Mc^2}{10^6\, GeV}\right)^{-2} \quad \leftarrow 10^5\, GeV$$

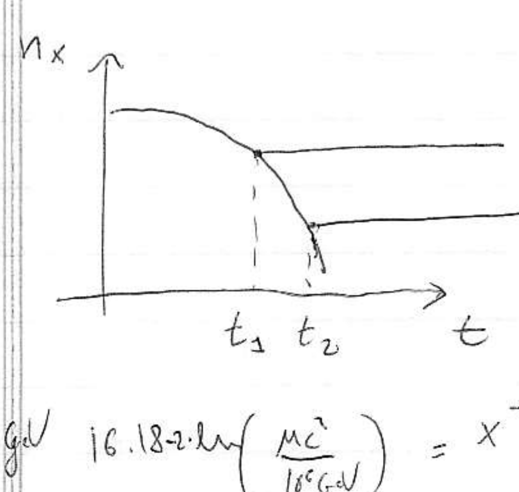
$$\ln F + \cancel{\tfrac{1}{2}\ln(\ln F)} \simeq \underbrace{\left(\frac{Mc^2}{10^6\, GeV}\right)^{-2}}_{= X \simeq 18} \Rightarrow F \simeq 1.4 \left(\frac{m_x c^2}{1\, GeV}\right)\left(\frac{\sigma_0}{10^{-44} cm^2}\right) \simeq$$

$$\simeq e^{(\quad)^{-2}}$$

$$2.15 \cdot 10^6 \left(\frac{m_x c^2}{1\, GeV}\right) \simeq \frac{e^X}{X}$$

$\simeq 2.15 \cdot 10^5\, GeV$

$m_x c^2 / GeV$	$c\,\sigma_0\ (10^{-26}\ cm^3/s)$	$n_x \propto \frac{1}{m_x}$
5	0.945	1
10	0.979	1/2
30	1.033	1/6
80	1.081	1/16



eg; $m_x c^2 = 5\, GeV$; $\frac{c\sigma_0}{10^{-26}} \sim 0.945$

eg; $m_x c^2 \simeq 80\, GeV$; $\frac{c\sigma_0}{10^{-26}} \sim 1.081$

$m_x c^2 = 5\, GeV$ $\quad 16.18 - 2 \cdot \ln \underbrace{\left(\frac{Mc^2}{10^6\, GeV}\right)}_{= X} = X^{-2}$

$$x \simeq (16.18 - 2\ln x)^{-\frac{1}{2}} \simeq 0.23$$

$x^0 = 0.24$

$x^1 = 0.23$