THEORY of STRUCTURE FORMATION

- **Jeans theory: grav. instability.**

$\nearrow$ grav. force > Pressure force

$$\frac{F_g}{M} = \frac{GM}{\lambda^2} = \frac{G\rho\lambda^3}{\lambda^2} > \frac{F_p}{M} = \frac{P\lambda^2}{\rho\lambda^3} = \frac{v_s^2}{\lambda}$$

sound speed

$$v_s^2 \simeq \gamma\frac{P}{\rho} \qquad \left(v_s^2 = \frac{dP}{d\rho}\right);\ P = k\rho^\gamma$$

$$\lambda > \frac{v_s}{(G\rho)^{\frac{1}{2}}} = \lambda_J \qquad t_{ff} = \frac{1}{(G\rho)^{\frac{1}{2}}} \qquad t_{cr} = \frac{\lambda}{v_s}$$

$$\boxed{t_{cr} > t_{ff}}$$

sound speed crossing time > free-fall time

- **Collisional fluid - perturbation analysis:**

Euler equations $\hookrightarrow$

$$\frac{\partial\rho}{\partial t} + \bar\nabla\cdot(\rho\vec v) = 0 \quad \leftarrow \text{mass conserv.}$$

momentum cons.

$$\frac{\partial\bar v}{\partial t} + v\cdot\bar\nabla\bar v + \frac{1}{\rho}\bar\nabla P + \nabla\Phi = 0$$

entropy cons.

$$\frac{DS}{Dt} + v\cdot\bar\nabla S = 0$$

$$\nabla^2\Phi = 4\pi G\rho \quad \text{Poisson eq.}$$

S = entropy per unit mass (derivative is lagrangian)

perturbation analysis

$$\rho = \rho_0 + \delta\rho \qquad v = \delta v \qquad P = P_0 + \delta P \qquad S = S_0 + \delta S$$

small perturbation

1st. order equations: $\frac{\partial \rho_0}{\partial t} = 0$ 2nd order (neglect)

$$\left\{\begin{array}{l} \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \delta v = 0 \\ \frac{\partial \delta v}{\partial t} + \frac{1}{\rho_0}\left(\frac{\partial P}{\partial \rho}\right)_s \nabla \delta\rho + \frac{1}{\rho_0}\left(\frac{\partial P}{\partial s}\right)_\rho \nabla \delta s + \nabla \delta\phi = 0 \\ \nabla^2 \delta\phi = 4\pi G \delta\rho \end{array}\right.$$

$$\left[\cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \delta\rho}{\partial t} + \vec{\nabla}[(\rho_0 + \cancel{\delta\rho})(\delta v)] = 0 \right]$$

$$\boxed{\frac{D\delta s}{Dt} = 0}$$

$$\cancel{\frac{\partial s_0}{\partial t}} + \frac{\partial \delta s}{\partial t} + \delta v \cdot \underbrace{\vec{\nabla}(s_0}_{0} + \cancel{\delta s}) = 0 \quad \text{2nd order}$$

$$v_s = \left(\frac{\partial P}{\partial \rho}\right)_s$$

Plane solutions: $\delta u_i = \delta_{i0} \exp(i\vec{k}\cdot\vec{r} + i\omega t)$

$i = 1,2,3,4$ $(\delta\rho, \delta\vec{v}, \delta\phi, \delta s)$

$\delta_{i0} = D \quad \vec{V} \quad \varphi \quad \Sigma$

$$\left\{\begin{array}{l} \omega \delta_0 + \vec{k}\vec{V} = 0 \\ \omega \vec{V} + k v_s^2 \delta_0 + \frac{\vec{k}}{\rho_0}\left(\frac{\partial P}{\partial s}\right)_\rho \Sigma + \vec{k}\varphi = 0 \\ k^2 \varphi + 4\pi G \rho_0 \delta_0 = 0 \\ \omega \Sigma = 0 \quad \rightarrow [\omega \neq 0 \rightarrow \Sigma = 0 \text{ adiabatic}] \end{array}\right.$$

$$\boxed{\delta_0 = \frac{D}{\rho_0}}$$

$\omega \neq 0$ time dependent solutions $\Rightarrow \vec{k}\vec{V} \neq 0$

($\vec{k} \parallel \vec{v}$ longitudinal perturbations) $\Rightarrow$ $\underline{\Sigma = 0 \text{ adiabatic perturbati}}$

$k \perp v \rightarrow$ vertical modes, stationary ($\omega = 0$), entropic) solution for δ_0, V and φ only if

$$\left\{\begin{array}{l} \omega \delta_0 + k V_\parallel = 0 \\ \omega V_\parallel + k v_s^2 \delta_0 + k\varphi = 0 \\ k^2 \varphi + 4\pi G \rho_0 \delta_0 = 0 \end{array}\right. \Longrightarrow \omega^2 - v_s^2 k^2 + 4\pi G \rho_0 = 0$$

$$\begin{vmatrix} \omega & k & 0 \\ k v^2 & \omega & k \\ 4\pi G \rho_0 & 0 & k^2 \end{vmatrix} = \omega\, \omega \cancel{k^2} + \cancel{k^2}(k^2 v^2 - 4\pi G \rho_0) = 0$$

$$\omega = \pm v_s k \left[1 - \left(\frac{\lambda}{\lambda_J}\right)^2\right]^{\frac{1}{2}}$$

where $\lambda_J = v_s \left(\frac{\pi}{G\rho_0}\right)^{\frac{1}{2}}$ Jeans length

$$\frac{\delta\rho}{\rho_0} = \delta_0 \exp[i(\bar{k}\bar{r} \pm |\omega| t)]$$

$$\delta v = \mp \frac{\vec{k}}{k} v_s \underbrace{\delta_0 \left[1-\left(\frac{\lambda}{\lambda_J}\right)^2\right]^{\frac{1}{2}} \exp[i(\bar{k}\bar{r} \pm |\omega| t)]}_{\to \frac{\delta\rho}{\rho_0}}$$

$$\delta\varphi = -\underbrace{\delta_0 v_s^2 \left(\frac{\lambda}{\lambda_J}\right)^2 \exp[i(\bar{k}\bar{r} \pm |\omega| t]}_{\to \frac{\delta\rho}{\rho_0}}$$

a) $\lambda < \lambda_J$ Sound waves in direction $\pm\vec{K}$ $\quad v_f = \frac{\omega}{k} = \pm v_s \left[1-\left(\frac{\lambda}{\lambda_J}\right)^2\right]^{\frac{1}{2}}$

v_f = phase velocity $= \frac{\omega}{K} \xrightarrow[\lambda\to\lambda_J]{} 0$

b) $\lambda > \lambda_J$ $\quad \omega = \pm i \underbrace{(4\pi G\rho_0)^{\frac{1}{2}}}_{= \frac{1}{t_{ff}}} \cdot \left[1 - \underbrace{\left(\frac{\lambda_J}{\lambda}\right)^2}_{<1}\right]^{\frac{1}{2}}$

(unstable solution)

$$\frac{\delta\rho}{\rho_0} = \delta_0 \exp(i\bar{k}\bar{r})\, e^{\pm|\omega| t}$$

etc..

exponential collapse or rarefaction with time scale: $\quad t = t_{ff}\left[1-\left(\frac{\lambda_J}{\lambda}\right)^2\right]^{\frac{1}{2}} \xrightarrow[\lambda \gg \lambda_J]{} t_{ff}$

(15) structure formation (2)

Collisionless fluid:

Vlasov, ~~Liouville~~ equation (Boltzmann eq. with 0 source term ... the left hand side)

(1) $\begin{cases} \dfrac{\partial f}{\partial t} + \vec{\nabla}_x f \, \vec{v} + \vec{\nabla}_v f \, \dot{\vec{v}} = 0 \\ \nabla^2 \varphi = 4\pi G \rho \;;\; \dot{\vec{v}} = \vec{g} = -\nabla \varphi \end{cases}$

$\nabla_{\vec{v}} = \dfrac{\partial}{\partial \vec{v}}$ subscript

$\nabla_{\vec{x}} = \dfrac{\partial}{\partial \vec{r}}$ subscript

$f = F(\vec{r}; \vec{v}; t)$ phase-space distribution $\left(\rho = m \int f \, dv\right)$

(2) $\nabla^2 \delta\varphi = 4\pi G \delta\rho$ Poisson equation

(3) $\delta\rho = m \cdot \int \delta f \, dv$ $\vec{g} = \dot{\vec{v}}$ $\delta\dot{\vec{v}} = \delta\vec{g} = -\nabla\delta\varphi$)

$\dfrac{\partial}{\partial t}\delta f + \vec{v}\,\nabla\delta f \underbrace{- \nabla\delta\varphi}_{\delta\dot{\vec{v}}} \nabla_{\vec{v}} f = 0$

$\dfrac{\partial}{\partial t} \to i\omega$ $\dfrac{\partial}{\partial x} \to ik$ solving we get: $\left(\dfrac{\partial}{\partial \vec{v}} \to v_x \dfrac{\partial}{\partial v^2}\right)$

(4) $\delta f = -4\pi G \dfrac{\partial f}{\partial v^2} \dfrac{v_x}{k(\omega + k v_x)} \cdot \delta\rho$ $i\omega\delta f + i v_x k \delta f = \dfrac{4\pi G \delta\rho}{ik} v_x \dfrac{\partial}{\partial v^2}$

using (3) we get:

(5) $\cancel{\delta\rho} = -m \int 4\pi G \dfrac{\partial f}{\partial v^2} \dfrac{v_x}{k(\omega + k v_x)} \, d\vec{v} \cdot \cancel{\delta\rho}$ symmetry

$\dfrac{\omega^2}{k} \cdot \left(k + \dfrac{4\pi G m}{\omega} \int \dfrac{v_x}{\left(1 + \frac{k v_x}{\omega}\right)} \dfrac{\partial f \, d\vec{v}}{\partial v^2} = 0 \right)$ $\omega^2 + 4\pi G m \dfrac{\omega}{k}\Bigg(\overbrace{\int v_x \dfrac{\partial f}{\partial v^2} d\vec{v}}^{0} +$

$\boxed{\text{if } k v_x \ll \omega}$ $\left(1 + \dfrac{k v_x}{\omega} + \dots\right)$ $\ll 1$ $+ \dfrac{k}{\omega}\int v_x^2 \dfrac{\partial f}{\partial v^2} d\vec{v} + \dots \Bigg) = 0$

$\omega^2 = -4\pi G m \overbrace{\int f \, dv}^{\rho_{11}}$ $f(v^2) \xrightarrow[v \to \infty]{} 0$ (integrate by parts)

$\left(d\vec{v} = dv_x^2 + dv_y^2 + dv_z^2\right)$ $v^2 f \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f \, d\vec{v}$

$$\boxed{\omega^2 = -4\pi G\rho} \qquad \left(\omega^2 \simeq \frac{1}{t_{ff}^2}\right) \quad t_{ff} = \sqrt{\frac{1}{4\pi G\rho}}$$ 59

general solution: $\omega^2 < 0$ unstable solution

$\tau = t_{ff}$ (time scale of instability)

Setting $\omega = 0$ in (5):

$$\lambda_J = v_* \left(\frac{\pi}{G\rho}\right)^{\frac{1}{2}} \qquad v_*^{-2} = \frac{\int v^{-2} f d^3\vec{v}}{\int f d^3\vec{v}} = \langle v_*^{-2} \rangle$$

if f is Maxwellian $f(v) = \frac{\rho}{(2\pi\sigma^2)^{3/2}} \exp\left(-\frac{v^2}{2\sigma^2}\right)$

$\Downarrow$

$$\boxed{v_* = \sigma}$$

Effect of cosmic expansion:

$$\frac{\partial \delta}{\partial t} = \pm \frac{\delta}{t_{ff}}$$ collapsing solution

$$\rho_m \simeq \frac{1}{6\pi G t_H^2} \qquad (t_{ff} \simeq t_H)$$

matter dominated

$$t_{ff}^2 = \frac{1}{4\pi G\rho} \rightarrow t_{ff} = \sqrt{\frac{3}{2}}\, t$$

$$\frac{\partial \delta}{\delta} = \pm\sqrt{\frac{2}{3}} \frac{\partial t}{t} \qquad \delta_\pm = A\, t^{\pm\sqrt{\frac{2}{3}}} \sim A\, t^{\pm 0.816}$$

But $A \neq$ const because during evolution δ and ρ_m are changing due to Hubble expansion

$$A \propto t^{-\frac{1}{6}} \propto t^{-0.166} \Longrightarrow \delta_\pm = \begin{cases} \delta_+ = t^{0.65} \simeq t^{\frac{2}{3}} \\ \delta_- = t^{-0.98} \simeq t^{-1} \end{cases}$$

collapsing solution

rarefaction solution

adiabatic invariants

(matter dominated)

$$\cancel{\rho_{rad} = \frac{3}{32\pi G t^2} \Longrightarrow t_{ff} = \sqrt{\frac{8}{3}}\, t}$$

$$\cancel{\delta_\pm = A\, t^{\pm\frac{1}{2}\sqrt{\frac{3}{2}}} \propto t^{\pm 0.61}}$$

$$\delta_\pm = \begin{cases} \delta_+ \propto t \\ \delta_- \propto t^{-1} \end{cases}$$ (radiation dominated)

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Adiabatic invariants:

$\nu_{oscill} >> \frac{1}{t_{var}}$ $\left(\nu_{oscill} >> \frac{1}{t_H} \right)$

Valid only for perturbations with $\lambda < \lambda_J$ (sound waves)

Then $\boxed{\frac{E_{oscill}}{\nu_{oscill}} = \text{const}}$

$$\frac{E_{oscill}}{V} = \left(\frac{1}{2} \rho \, \delta v^2 + \frac{1}{2} v_s^2 \frac{\delta\rho^2}{\rho} \right) = \frac{1}{2}\left(v_s^2 \frac{\delta\rho^2}{\rho} + v_s^2 \frac{\delta\rho^2}{\rho} \right) = v_s^2 \frac{\delta\rho^2}{\rho}$$

$$\left(\begin{array}{l} \delta v = \frac{\delta\rho}{\rho} v_s \\ \delta v^2 = \frac{\delta\rho^2}{\rho^2} v_s^2 \end{array} \right) \qquad = \delta^2 \rho v_s^2$$

$t_{osc} = \nu^{-1} = \frac{\lambda}{v_s}$ $\qquad \frac{E_{osc}}{\nu} \propto v_s \lambda \frac{\delta\rho^2}{\rho} = \text{const}$

$P = P_r + \cancel{P_m}^{\,0} \propto \rho_r \propto \rho_m^{\frac{4}{3}}$ $\qquad \frac{\delta\rho^2}{\rho^2} = \delta^2 = \frac{1}{\rho \, v_s \lambda}$

$v_s = \left(\frac{P_r}{\rho_m} \right)^{\frac{1}{2}} \propto \rho_m^{1/6} \propto a^{-\frac{1}{2}} \propto t^{-\frac{1}{3}}$ $\quad (\rho_m \propto a^{-3})$

$\delta^2 \propto \frac{\delta\rho^2}{\rho^2} \propto \rho_m^{-\frac{1}{6}-1}$ $\quad (\rho_m \propto t^{-2})$ $\quad ?$ $\underline{\delta \propto t^{-\frac{1}{6}}}$ $\;?$

$\boxed{\text{SKIP}}$

Newtonian theory in a Dust Universe:

$$\begin{cases} \rho = \rho_0 \left(\frac{a}{a_0}\right)^{-3} & P = P(\rho; S) \\ \vec{v} = H\vec{r} & S = \text{const} \\ \varphi = \frac{2}{3}\pi G \rho r^2 \end{cases}$$

$$\vec{r} = \vec{r}_{com.} \frac{a}{a_0}$$

Note: Newtonian theory not consistent because for $r \to \infty$ v and $\varphi \to \infty$. Relativistic treatment removes these problems.

Small perturbations: $\delta\rho$; δv; $\delta\varphi$ and δp

$$\begin{cases} \delta\dot{\rho} + 3\overset{H}{\frac{\dot{a}}{a}}\delta\rho + \cancel{\frac{\dot{a}}{a}(\vec{r}\cdot\vec{\nabla})\delta\rho}^{H\vec{r} = \vec{v} = 0} + \rho(\vec{\nabla}\cdot\delta\vec{v}) = 0 \\ \delta\dot{v} + H\delta v + \underbrace{\cancel{H(\vec{r}\vec{\nabla})\delta v}}_{\to \vec{v} = 0} = -\frac{1}{\rho}\vec{\nabla}\delta P - \vec{\nabla}\delta\varphi \\ \nabla^2\delta\varphi = 4\pi G\delta\rho \\ \delta\dot{S} + \underbrace{\cancel{H(\vec{r}\vec{\nabla})\delta S}}_{\vec{v} = 0} = 0 \end{cases}$$

$\Downarrow$ assume $\vec{v} = 0$; (1) (2) $\boxed{D = \frac{\delta\rho}{\rho};\ \vec{V} = \delta\vec{v}}$

Plane waves: $\delta x = \underbrace{X(t)}\exp(i\vec{k}\vec{x})$ $\boxed{\phi = \delta\varphi;\ \Sigma = \delta S}$

amplitude of plane waves

$$\begin{cases} \dot{D} + 3HD + i\rho\vec{k}\vec{V} = 0 \\ \dot{\vec{V}} + H\vec{V} + i v_s^2 k \frac{D}{\rho} + \frac{i\vec{k}}{\rho}\left(\frac{\partial P}{\partial S}\right)_\rho \Sigma + i\vec{k}\phi = 0 \\ k^2\phi + 4\pi G D = 0 \\ \dot{\Sigma} = 0 \end{cases}$$

a) Time independent solution : $\Sigma = \phi = D = 0 \; + \vec{V}\cdot\vec{k} = 0$

X

$\dot{V} + HV = 0 \quad \rightarrow \quad V = V_0 \cdot \frac{a_0}{a}$ (vortical solution: $\vec{V}_0 \perp \vec{k}$)

b) $\Sigma = 0 \qquad \vec{V}\cdot\vec{k} = \underset{V}{|\vec{V}|}\underset{k}{|\vec{k}|} \qquad \vec{V} \parallel \vec{k}$

$$\begin{cases} \dot{D} + 3HD + i\rho kV = 0 \\ \dot{V} + HV + ik\left(v_s^2 - \frac{4\pi G\rho}{k^2}\right)\frac{D}{\rho} = 0 \end{cases}$$

$D = \delta \cdot \rho$ ↳ perturbed density

$\dot{D} = \frac{d}{dt}(\rho\delta) = \rho\dot{\delta} + \delta(3H\rho) = \rho(\dot{\delta} - 3H\delta)$

$\left(\begin{array}{l} \rho \propto a^{-3} \quad \ln\rho = -3\ln a \\ \frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a} = -3H \end{array} \right)$

$\rho\dot{\delta} \cancel{-3HD} \cancel{+3HD} + ik\rho V = 0 \;\Rightarrow\; \dot{\delta} + ikV = 0$

differentiating with respect to time: $\left(k \propto \frac{1}{a}\right)$

$\ddot{\delta} + ik\dot{V} - ik\overset{H}{\frac{\dot{a}}{a}}V = 0 \;\rightarrow\; \ddot{\delta} + ik(\dot{V} - HV) = 0$

finally substituting expression for $\dot{V}$ and V:

$\ddot{\delta} + ik\left[-2HV - ik\left(v_s^2 - \frac{4\pi G\rho}{k^2}\right)\delta\right] = \ddot{\delta} - 2H\overset{-\dot{\delta}}{(ikV)} + k^2\Big($

$\left. v_s^2 - \frac{4\pi G\rho}{k^2}\right)\delta = \ddot{\delta} + 2H\dot{\delta} + (v_s^2k^2 - 4\pi G\rho)\delta = 0$

$k_J = \frac{2\pi}{\lambda_J} = \frac{2\sqrt{\pi G\rho}}{v_s} \quad \rightarrow \quad \lambda_J = v_s\sqrt{\frac{\pi}{G\rho}}$

$$\boxed{\ddot{\delta} + 2H\dot{\delta} + v_s^2(k^2 - k_J^2)\delta = 0}$$

Solution for a flat universe:

$$\begin{cases} \rho = \dfrac{1}{6\pi G t^2} \\ a = a_0 \left(\dfrac{t}{t_0}\right)^{\frac{2}{3}} \\ H = \dfrac{2}{3}\dfrac{1}{t} \end{cases}$$

dust-model $(w=0)$
|||
matter dominated

$$4\pi G\rho = \frac{2}{3}\frac{1}{t^2}$$

$$\ddot{\delta} + \frac{4}{3}\frac{\dot{\delta}}{t} - \frac{2}{3}\frac{1}{t^2}\left(1 - \frac{k^2}{k_J^2}\right)\delta = 0$$

for $k \to 0$ $\left(\frac{k}{k_J} \ll 1 \text{ or } \lambda \gg \lambda_J\right)$

$$\ddot{\delta} + \frac{4}{3}\frac{\dot{\delta}}{t} - \frac{2}{3}\frac{\delta}{t^2} = 0 \qquad \delta \propto t^n \qquad \dot{\delta} \propto n t^{n-1}$$

$$\ddot{\delta} \propto n(n-1) t^{n-2}$$

$$n(n-1)\cancel{t^{n-2}} + \frac{4}{3} n \cancel{t^{n-2}} - \frac{2}{3}\cancel{t^{n-2}} = 0$$

$$n^2 - n + \frac{4}{3}n - \frac{2}{3} = 0 \qquad n^2 + \frac{1}{3}n - \frac{2}{3} = 0 \qquad n = \frac{-1 \pm \sqrt{1+24}}{6} = \begin{cases} +\frac{2}{3} \\ -1 \end{cases}$$

$$\delta_{\pm} = \begin{cases} \delta_+ \propto t^{\frac{2}{3}} \propto a \\ \delta_- \propto t^{-1} \propto a^{-1.5} \end{cases} \quad \text{for } \lambda \gg \lambda_J \quad \text{collapsing pert.}$$

Solution for a radiation-dominated universe:

$$\rho \rightarrow \rho + \frac{3P}{c^2} = 2\rho \quad \text{for } P = \frac{1}{3}\rho c^2 \quad \text{pure rad dominated}$$

$$\frac{\dot{\rho}}{\rho} = -3H \underset{(\rho \propto a^{-3})}{} \longrightarrow \frac{\dot{\rho}}{\rho} = -4H \quad (\rho \propto a^{-4}) \qquad k_J^2 \rightarrow \left(\frac{4}{3} \cdot 2\right) \times 4\pi \ldots$$

$$\ddot{\delta} + 2H\dot{\delta} + v_s^2\left(k^2 - \frac{32}{3}\frac{\pi G\rho}{v_s^2}\right) = 0 \qquad \boxed{\text{where } v_s = \frac{c}{\sqrt{3}}}$$

$$k_J = \frac{2\pi}{\lambda_J} = 4\pi\sqrt{\frac{32\,\cancel{3}\,G\rho}{\pi c^2}} \implies \lambda_J = \frac{c}{2}\sqrt{\frac{\pi\,\cancel{3}}{2G\rho}} = v_s\sqrt{\frac{3\pi}{8G\rho}}$$

$$\begin{cases} \rho = \dfrac{3}{32\pi G t^2} \\ a = a_0\left(\dfrac{t}{t_{eq}}\right)^{\frac{1}{2}} \\ H = \dfrac{1}{2}\dfrac{1}{t} \end{cases} \quad \text{rad. dominated}$$

$$\frac{32}{3}\pi G\rho = \frac{1}{t^2}$$

$$\ddot{\delta} + \frac{\dot{\delta}}{t} - \frac{1}{t^2}\left(1 - \frac{k^2}{k_J^2}\right)\delta = 0$$

for $\frac{k}{k_J} << 1$ $\quad \lambda >> \lambda_J$ we have $\quad \delta \propto t^n$

$$\ddot{\delta} + \frac{\dot{\delta}}{t} - \frac{\delta}{t^2} = 0 \implies n(n-1) + n - 1 = 0$$

$$n^2 - 1 = 0 \qquad n = \pm 1 \qquad \delta_\pm = \begin{cases} \delta_+ \propto t \propto a^2 \\ \delta_- \propto t^{-1} \propto a^{-2} \end{cases}$$

In general $P = w\rho c^2$:
$$\delta_\pm = \begin{cases} \delta_+ \propto t^{\frac{2(1+3w)}{3(1+w)}} \\ \delta_- \propto t^{-1} \end{cases}$$

$$\lambda_J = \frac{\sqrt{24}}{5+9w} v_s \left(\frac{\pi}{G\rho}\right)^{\frac{1}{2}}$$

Multi fluid equations:

~~ii) The Meszaros Effect?~~

relativistic component $\rho_r \propto a^{-4}$ + non-relativistic component $\rho_m \propto a^{-3}$.

a) relativistic perturbations $= 0$ $\left\{ \begin{array}{l} \text{isothermal pert.} \\ \text{(if baryon dominated)} \\ \text{isocurvature pert.} \\ \text{(if dark matter dominated)} \end{array} \right.$

b) isentropic perturbations: both rel. and non-relat perturbations are $\neq 0$.

Meszaros effect: matter-perturbations are frozen before z_{eq} if relativistic perturbations are negligible (isothermal perturbations or isocurvature)

$z \gg z_{eq}$: $t_{ff} = \frac{1}{\sqrt{G\rho_m}}$ $\qquad t_H = \frac{1}{\sqrt{G\rho_{rad}}}$

Before z_{eq} $\rho_{rad} > \rho_m \implies t_{ff} > t_H$

matter-pert are frozen

$z < z_{eq}$ $\quad t_{ff} = \frac{1}{\sqrt{G\rho_m}}$ $\quad t_H = \frac{1}{\sqrt{G\rho_m}}$ $\qquad t_{ff} \sim t_H$

matter-pert can grow

$y = \frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}}$; $\ddot{\delta} + 2H\dot{\delta} - 4\pi G \rho_m \delta = 0$ (for $\lambda \gg \lambda_J$)

solution $\delta_+ \propto 1 + \frac{3}{2} y \implies \frac{\delta_+(y=1)}{\delta_+(y=0)} = \frac{5}{2}$

$\delta_+(y \gg 1) \propto y \propto a \propto t^{\frac{2}{3}}$

Two Fluid Models: (I) Baryonic matter + photons.

1) Adiabatic & Isothermal perturbations:

We have seen that perturbations can be "vortical" ($\vec{k} \perp \vec{v}$) or longitudinal ($\vec{k} \parallel \vec{v}$) entropic ($\Sigma \neq 0$) or adiabatic ($\Sigma = 0$)

a) $$S = \underbrace{\frac{4}{3}\sigma T^3}_{s} \cdot V \propto \frac{T^3}{\rho_m} \propto \frac{\rho_r^{\frac{3}{4}}}{\rho_m} \qquad \left(V \propto \frac{1}{\rho_m} \,;\, \rho_r \propto T^4 \right)$$

$$\frac{\delta S}{S} = \frac{3}{4} \overbrace{\frac{\delta \rho_r}{\rho_r}}^{\delta_r} - \overbrace{\frac{\delta \rho_m}{\rho_m}}^{\delta_m} = 0 \qquad \left(\text{adiabatic perturbations} \right)$$

$$\boxed{\delta_m = \frac{3}{4}\delta_r = 3\frac{\delta T}{T}} \qquad \text{ADIABATIC}$$

b) ISOTHERMAL : $\boxed{\delta_m \neq 0 \;;\; \delta_r = 0}$ $(\delta S \neq 0)$

2) Sound speed & Jeans mass:

adiabatic: $v_s^{(a)} = \left(\dfrac{\partial P}{\partial \rho}\right)_S^{\frac{1}{2}}$ $\qquad \left| \begin{array}{l} P = P_r + P_m \simeq P_r \simeq \frac{1}{3}\rho_r c^2 \\ \rho = \rho_m + \rho_r \end{array} \right.$

$$v_s^{(a)} = \frac{c}{\sqrt{3}}\left[1 + \left(\frac{\partial \rho_m}{\partial \rho_r}\right)_S\right]^{-\frac{1}{2}} = \frac{c}{\sqrt{3}}\left(1 + \underbrace{\frac{3}{4}\frac{\rho_m}{\rho_r}}_{\simeq 1.2\left(\frac{3}{4}K_0\right)\left(\frac{1+z_{eq}}{1+z}\right)}\right)^{-\frac{1}{2}} \rightarrow \rho_r = \rho_\gamma$$

(do not include neutrinos because decoupled

$$v_s^{(a)} \simeq \begin{cases} \frac{c}{\sqrt{3}}\left(\frac{1+z}{1+z_{eq}}\right)^{\frac{1}{2}} & z < z_{eq} \\ \frac{c}{\sqrt{3}} & z > z_{eq} \end{cases}$$

isothermal: $v_s^{(i)} = \left(\frac{\partial P_m}{\partial \rho_m}\right)_S^{\frac{1}{2}} = \left(\gamma \frac{kT_m}{m_p}\right)^{\frac{1}{2}}$

$T_m = T_r = T_{r0}(1+z)$ for $z > z_{dec}$ $\quad \gamma = \frac{5}{3}$ monoatomic gas.

$$v_s^{(i)} = \left(\frac{\gamma k T_{rec}}{m_p}\right)^{\frac{1}{2}} \left(\frac{1+z}{1+z_{rec}}\right)^{\frac{1}{2}}$$

$M_J = \frac{1}{6}\pi \rho_m \lambda_J^3$; $\lambda_J = v_s\left(\frac{\pi}{G\rho_m}\right)^{\frac{1}{2}}$ Jeans Mass

→ only matter collapse (no-radiation)

$$M_J^{(a)} = \frac{1}{6}\pi \rho_m v_s^3 \left(\frac{\pi}{G\rho_m}\right)^{\frac{3}{2}}$$

$$M_J^{(a)} \propto \begin{cases} \rho_m^{-\frac{1}{2}}\left(\frac{1+z}{1+z_{eq}}\right)^{\frac{3}{2}} \propto \text{const} = M_{J,eq}^{(a)} & z < z_{eq} \\ \rho_m^{-\frac{1}{2}} = M_{J,eq}^{(a)}\left(\frac{1+z}{1+z_{eq}}\right)^{-\frac{3}{2}} & z > z_{eq} \end{cases}$$

$\left[M_{J,eq}^{(a)} = (3.5\cdot 10^{15}\, M_\odot)(\Omega h^2)^{-2}\right]$

$M_J^{(i)} \propto \rho_m^{-\frac{1}{2}}\left(\frac{1+z}{1+z_{rec}}\right)^{\frac{3}{2}} \propto \text{const}$ $\left[M_J^{(i)} = 5\cdot 10^4\, M_\odot (\Omega h^2)^{-\frac{1}{2}}\right]$

Evolution of Horizon Mass:

$M_H = \frac{1}{6}\pi \rho_m R_H^3$ $\quad R_H$ = horizon

↳ analogous definition to Jeans Mass

$z > z_{eq}$ $\quad \rho = \rho_r \Rightarrow M_H = \frac{1}{6}\pi \rho_m (2ct)^3 = M_H(z_{eq}) \left(\frac{1+z}{1+z_{eq}}\right)^{-3}$

$R_H = 2ct$ $\quad (1+z)^3$

$t \propto (1+z)^{-2}$

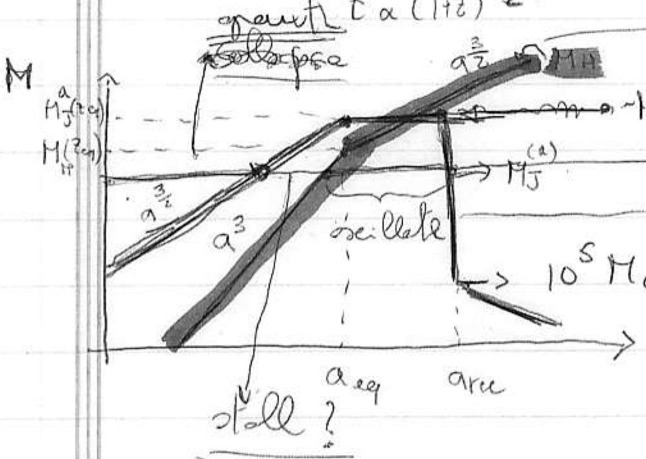
$M_H(z_{eq}) = 5 \cdot 10^{14} (\Omega_m h^2)^{-2} M_\odot \lesssim M_J^{(a)}(z_{eq})$

$z < z_{eq}$ $\quad \rho = \rho_m \Rightarrow M_H = \frac{1}{6}\pi \rho_m (3ct)^3 = M_H(z_{eq}) \left(\frac{1+z}{1+z_{eq}}\right)^{-\frac{3}{2}}$

$R_H = 3ct$ $\quad (1+z)^3$

$t \propto (1+z)^{-\frac{3}{2}}$

$M_{Hor} \propto a^2$ (rel. matter @ $z > z_{eq}$)



$\rightarrow M_{Horizon}$ (non-rel matter only)

$\rightarrow M_{Jeans}^{(adiab)}$ (non-relativistic matter only!)

1) $M > M_J^{(a)} \sim 10^{15} M_\odot$ uninterrupted growth $\begin{cases} \delta_m \propto a^2 & z > z_{eq} \\ \delta_m \propto a & z < z_{eq} \end{cases}$

2) $M_J^{(a)} \sim 10^{15} M_\odot < M < M_D^{(a)}(z_{rec}) \approx 10^{12} M_\odot$ oscillate between z_{eq}-z_{dec}

3) $M < M_D^{(a)}(z_{rec})$ damped by photon diffusion

(see next section)

$\Longleftrightarrow$ $M_D^{(a)}$ = mass scale for photon diffusion damping (Silk-damping)

$$\begin{cases} M_D^{(a)}(z_{eq}) = 7 \cdot 10^{10} M_\odot (\Omega h^2)^{-5} \left(\frac{1+z}{1+z_{eq}}\right)^{-\frac{9}{2}} \\ M_D^{(a)}(z_{rec}) = 1.7 \cdot 10^{14} M_\odot \left(\frac{\Omega_m h^2}{0.127}\right)^{-\frac{5}{2}} \approx 1.4 \cdot 10^{16} M_\odot \left(\frac{\Omega_m h^2}{0.022}\right)^{-\frac{5}{2}} \end{cases}$$

Dissipation of adiabatic perturbations:

In a fluid of electrons and photons, photon diffusion damps adiabatic perturbations

(Silk - damping).

The mass scale of damping is $M_D^{(a)} \simeq 10^{16} M_\odot$,

the scale of a cluster of galaxies

⇒ In a baryonic dominated universe structure with mass $< M_D^{(a)}$ cannot form.

Compton ~~drag~~ Drag:

Isothermal perturbations cannot grow due to friction force between e^- and γ:

⇒ perturbations with $M > M_J^{(i)}$ are frozen at $z_{eq} < z < z_{rec}$.

In summary without dark matter perturbations on scales smaller than a cluster of galaxies would not exist.

Standard Model:

"Cold" dark matter + gas + photons:

→ DM growth is slow during the period in which the photon-gas fluid perturbations oscillate and are damped. This is due to the Meszaros effect.

SEE plots for the evolution of $\delta_*; \delta_b; \delta_\gamma$ as a function of K or $[M(k)]$.

Transfer function:

$$T(K) = \frac{\delta_x(K; t=t_0)}{\delta_x(K\to 0; t=t_0)}$$

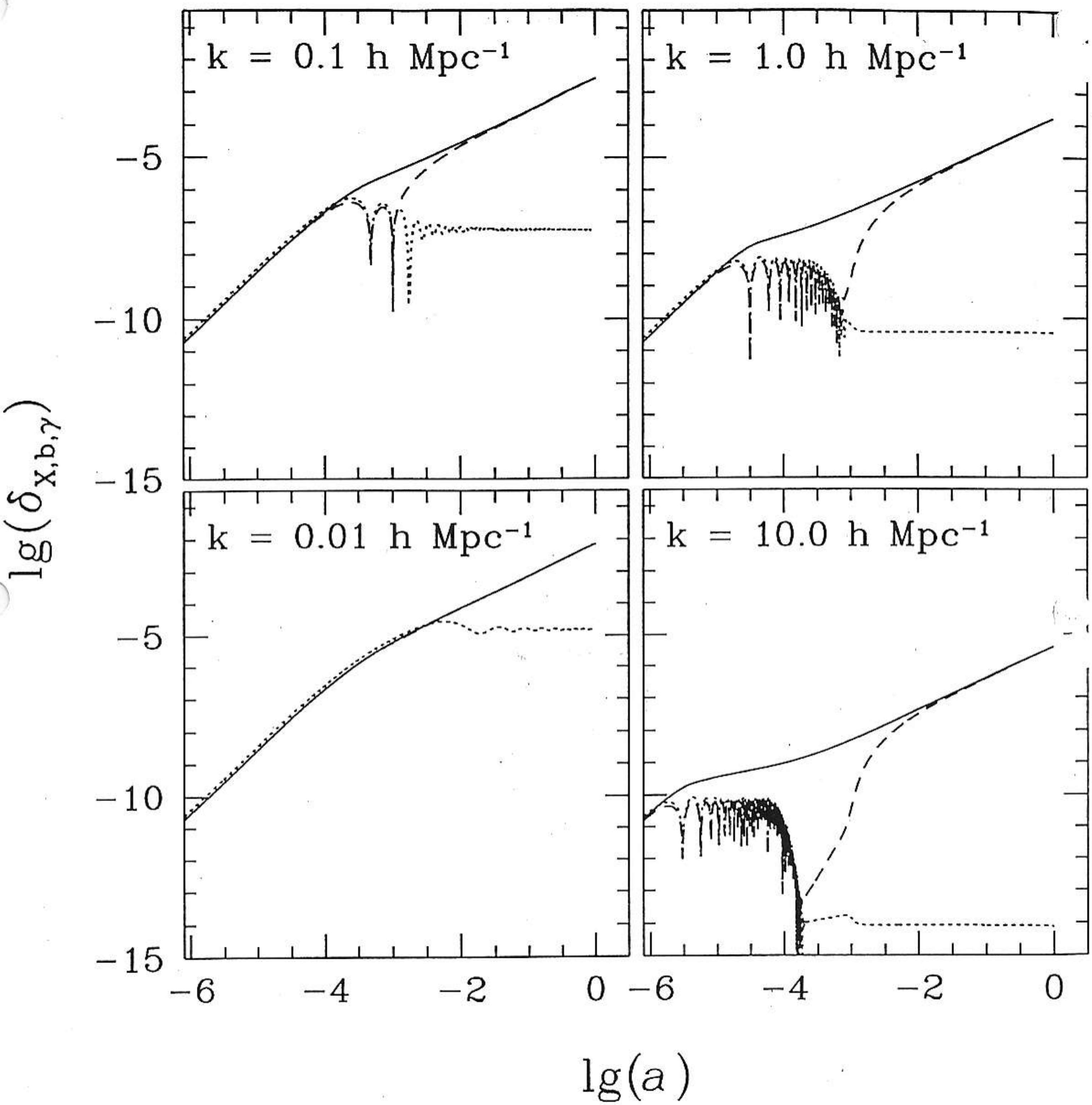
$$P(k) = P(k)_{\text{inflation}} \cdot T(k)^2 \qquad P(k) = \langle \tilde{\delta}_x^2 \rangle$$

$$P_{\text{inflation}} = k^n \qquad n \simeq 1 \pm \epsilon \qquad \text{WMAP } n = 0.95 \pm \ldots$$

$$T(k) \simeq \begin{cases} 1 & k \ll k_{eq} \\ \left(\frac{k_{eq}}{k}\right)^2 \left[1 + \ln\left(\frac{k}{k_{eq}}\right)\right] & k \gg k_{eq} \end{cases}$$

Exact solution: Bardeen et al. 1986 ApJ, 304, 15
(BBKS)

(217B)



k = 0.1 h Mpc⁻¹
k = 1.0 h Mpc⁻¹
k = 0.01 h Mpc⁻¹
k = 10.0 h Mpc⁻¹
−5
−10
−15
−6
−4
−2
0
lg(δ_{X,b,γ})
lg(a)

$g_{ik} = g^{0}_{ik} + h_{ik}$ $\quad h = \mathrm{Tr}(h_{ik})$ $\quad h_{33} = \vec{k} \parallel \hat{e}_z$ $\quad e^{i\vec{k}\vec{x}}$

(199A)

Metric

$$\begin{cases} \dfrac{d^2h}{dt^2} + 2H\dfrac{dh}{dt} = 3H_0^2\left(\dfrac{\Omega_X}{a^3}\delta_X + \dfrac{\Omega_B}{a^3}\delta_B + 2\dfrac{\Omega_\gamma}{a^4}\Delta_0 + 2\dfrac{\Omega_\nu}{a^4}\Xi_0\right) \\ \dfrac{dh_{33}}{dt} = \dfrac{dh}{dt} + 6H_0^2\dfrac{a}{kc}\left(\dfrac{\Omega_X}{a^3}u_X + \dfrac{\Omega_B}{a^3}u_B - \dfrac{\Omega_\gamma}{a^4}\Delta_1 - \dfrac{\Omega_\nu}{a^4}\Xi_1\right) \end{cases}$$

Newtonian reference frame

Dark matter

$$\begin{cases} \dfrac{d\delta_X}{dt} = \dfrac{1}{2}\dfrac{dh}{dt} + \dfrac{kc}{a}u_X \\ \dfrac{du_X}{dt} = -Hu_X \end{cases}$$

$$\begin{cases} \dfrac{\partial \delta_x}{dt} = -\mathrm{div}(\vec{V}_x) & \dfrac{\partial \delta_x}{dt} = -i\vec{k}\vec{V}_x \\ \dfrac{d\vec{V}_x}{dt} + H\vec{V}_x = -\dfrac{\nabla\varphi}{a^2} & \dfrac{d\vec{V}_x}{dt} + H\vec{V}_x = -\dfrac{i\vec{k}\varphi}{a^2} \end{cases}$$

$i\vec{V}_x = \frac{c}{a}u_x + \frac{1}{2k}\frac{dh}{dt} \qquad \varphi = f(H; \dots)$

Cosmic gas

$$\begin{cases} \dfrac{d\delta_B}{dt} = \dfrac{1}{2}\dfrac{dh}{dt} + \dfrac{kc}{a}u_B \\ \dfrac{du_B}{dt} = -Hu_B - c\sigma_T n_e\dfrac{\Omega_\gamma}{a\Omega_B}\left(\Delta_1 + \dfrac{4}{3}u_B\right) - \dfrac{k}{ac}c_S^2\delta_B \end{cases}$$

Compton scattering → $c\sigma_T n_e \, (u_\gamma - u_B)$; Pressure

Photon intensity

$$\begin{cases} \dfrac{d\Delta_0}{dt} = \dfrac{2}{3}\dfrac{dh}{dt} - \dfrac{kc}{a}\Delta_1 \\ \dfrac{d\Delta_1}{dt} = -c\sigma_T n_e(\Delta_1 + \dfrac{4}{3}u_B) + \dfrac{kc}{a}\left(\dfrac{1}{3}\Delta_0 - \dfrac{2}{3}\Delta_2\right) \\ \dfrac{d\Delta_2}{dt} = \dfrac{1}{10}c\sigma_T n_e\left(\Pi_0 + \Pi_2 - 9\Delta_2\right) + \dfrac{kc}{a}\left(\dfrac{2}{5}\Delta_1 - \dfrac{3}{5}\Delta_3\right) - \dfrac{2}{5}\left(\dfrac{dh_{33}}{dt} - \dfrac{1}{3}\dfrac{dh}{dt}\right. \\ \dfrac{d\Delta_l}{dt} = -c\sigma_T n_e\Delta_l + \dfrac{kc}{a}\left(\dfrac{l}{2l+1}\Delta_{l-1} - \dfrac{l+1}{2l+1}\Delta_{l+1}\right) \quad (l > 2) \end{cases}$$

$\Delta_0 \equiv$ energy

$\Delta_1 \equiv$ velocity $= -\frac{4}{3}u_\gamma$

Photon polarization (2 polarizations)

$$\begin{cases} \dfrac{d\Pi_0}{dt} = \dfrac{1}{2}c\sigma_T n_e\left(\Delta_2 + \Pi_2 - \Pi_0\right) - \dfrac{kc}{a}\Pi_1 \\ \dfrac{d\Pi_1}{dt} = -c\sigma_T n_e\Pi_1 + \dfrac{kc}{a}\left(\dfrac{1}{3}\Pi_0 - \dfrac{2}{3}\Pi_2\right) \\ \dfrac{d\Pi_2}{dt} = \dfrac{1}{10}c\sigma_T n_e\left(\Pi_0 + \Delta_2 - 9\Pi_2\right) + \dfrac{kc}{a}\left(\dfrac{2}{5}\Pi_1 - \dfrac{3}{5}\Pi_3\right) \\ \dfrac{d\Pi_l}{dt} = -c\sigma_T n_e\Pi_l + \dfrac{kc}{a}\left(\dfrac{l}{2l+1}\Pi_{l-1} - \dfrac{l+1}{2l+1}\Pi_{l+1}\right) \quad (l > 2) \end{cases}$$

Π = difference between 2 polarizations.

Compton scattering separate ($\Pi \neq 0$) right and left handed polarizations.

{ Compton scattering depends on polarization of the photons!

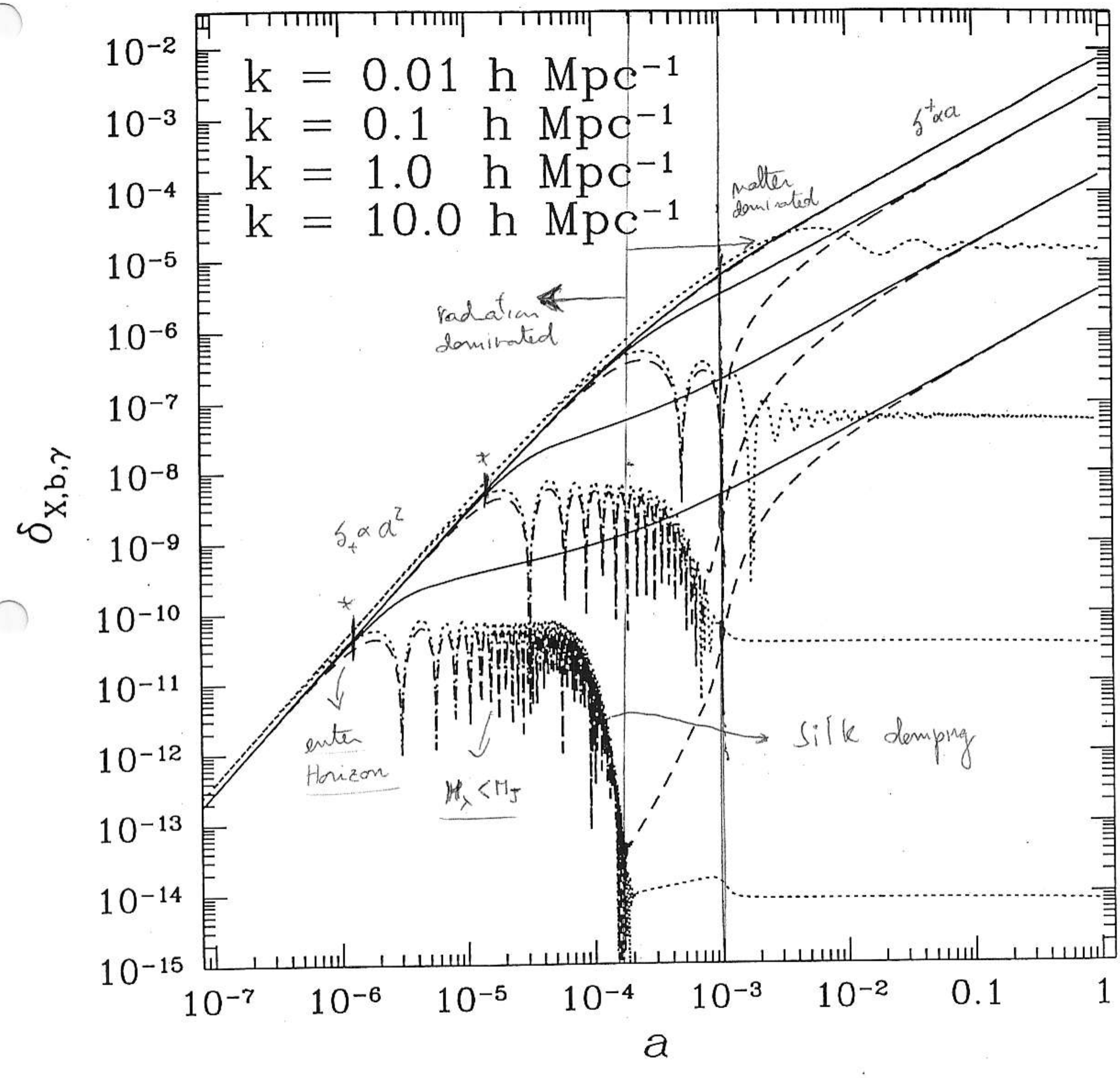
Massless neutrino

$$\begin{cases} \dfrac{d\Xi_0}{dt} = \dfrac{2}{3}\dfrac{dh}{dt} - \dfrac{kc}{a}\Xi_1 \\ \dfrac{d\Xi_1}{dt} = \dfrac{kc}{a}\left(\dfrac{1}{3}\Xi_0 - \dfrac{2}{3}\Xi_2\right) \\ \dfrac{d\Xi_2}{dt} = \dfrac{kc}{a}\left(\dfrac{2}{5}\Xi_1 - \dfrac{3}{5}\Xi_3\right) - \dfrac{2}{5}\left(\dfrac{dh_{33}}{dt} - \dfrac{1}{3}\dfrac{dh}{dt}\right) \\ \dfrac{d\Xi_l}{dt} = \dfrac{kc}{a}\left(\dfrac{l}{2l+1}\Xi_{l-1} - \dfrac{l+1}{2l+1}\Xi_{l+1}\right) \quad (l > 2) \end{cases}$$

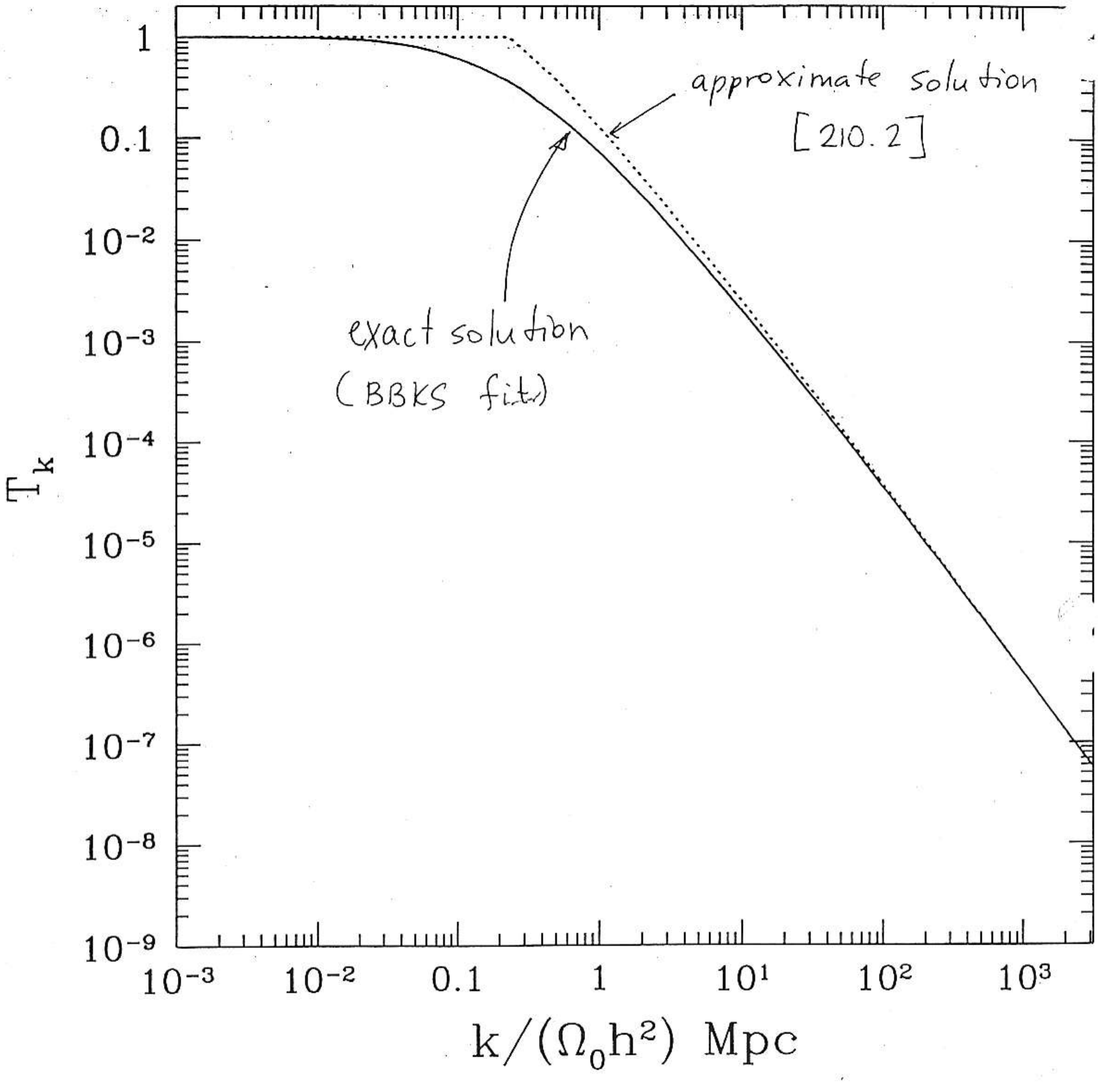
← (not interact with photons;} if massless!).

plus equations for a and thermal and ionization balance (n_e, c_S)

217A



k = 0.01 h Mpc^{-1}
k = 0.1 h Mpc^{-1}
k = 1.0 h Mpc^{-1}
k = 10.0 h Mpc^{-1}
$\delta^+ \propto a$
matter dominated
radiation dominated
$\delta_+ \propto a^2$
enter Horizon
$M_\lambda < M_J$
Silk damping
$\delta_{X,b,\gamma}$
a
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{-15}
10^{-7}
10^{-6}
10^{-5}
10^{-4}
10^{-3}
10^{-2}
0.1
1



$T_k \propto k^{-2} \cdot \lg k \qquad k \gg k_{eq}$

$P_k \propto k |T_k|^2 \propto \begin{cases} k \\ k \cdot k^{-4} \cdot (\lg k)^2 \propto k^{-3} (\lg k)^2 \end{cases}$

70

Hot dark matter:

Let's consider massive neutrinos as an example

$$\Omega_\nu h^2 = \frac{m_\nu c^2}{100\,\mathrm{eV}}$$

Neutrinos decouple when relativistic but become non-relativistic at:

$$k_B T_\nu \simeq m_\nu c^2 \qquad k_B \frac{1.9\,\mathrm{K}}{a} = m_\nu c^2 = \Omega_\nu h^2 \cdot 100\,\mathrm{eV}$$

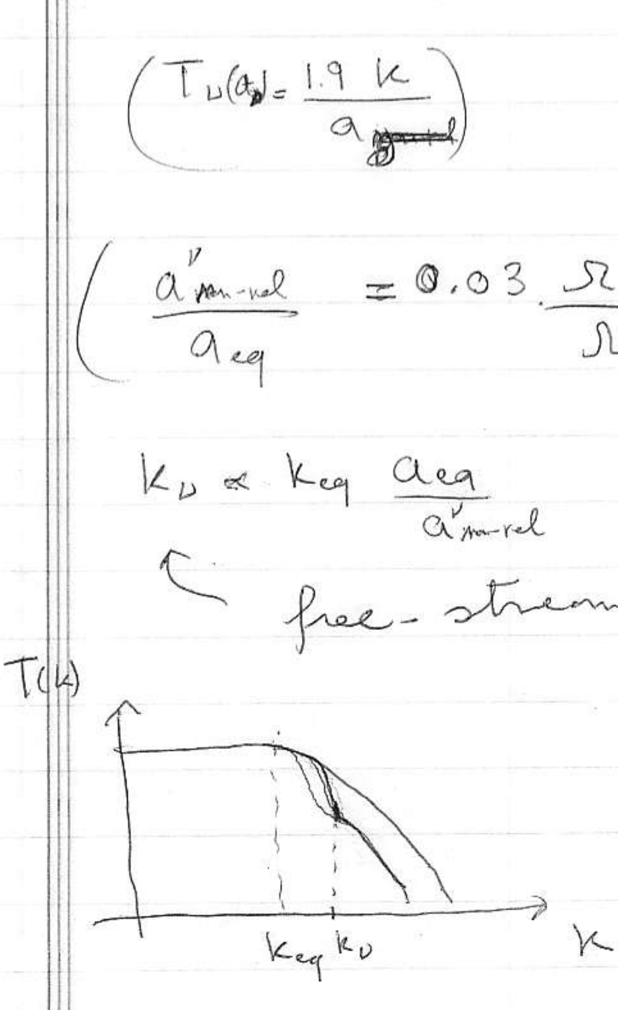
$$\left(T_\nu(a) = \frac{1.9\,\mathrm{K}}{a} \right) \qquad \Downarrow$$

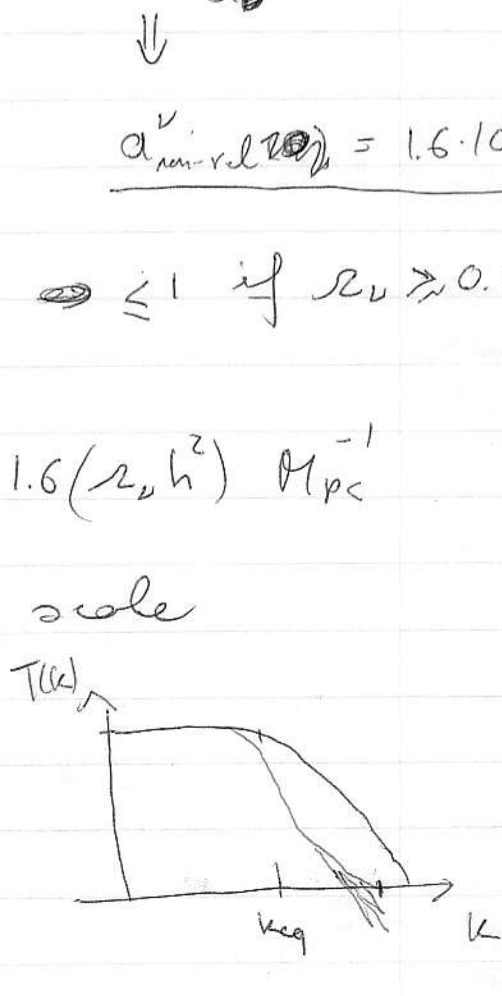
$$\underline{a^\nu_{\text{non-rel}} = 1.6 \cdot 10^{-6} \left(\Omega_\nu h^2\right)^{-1}}$$

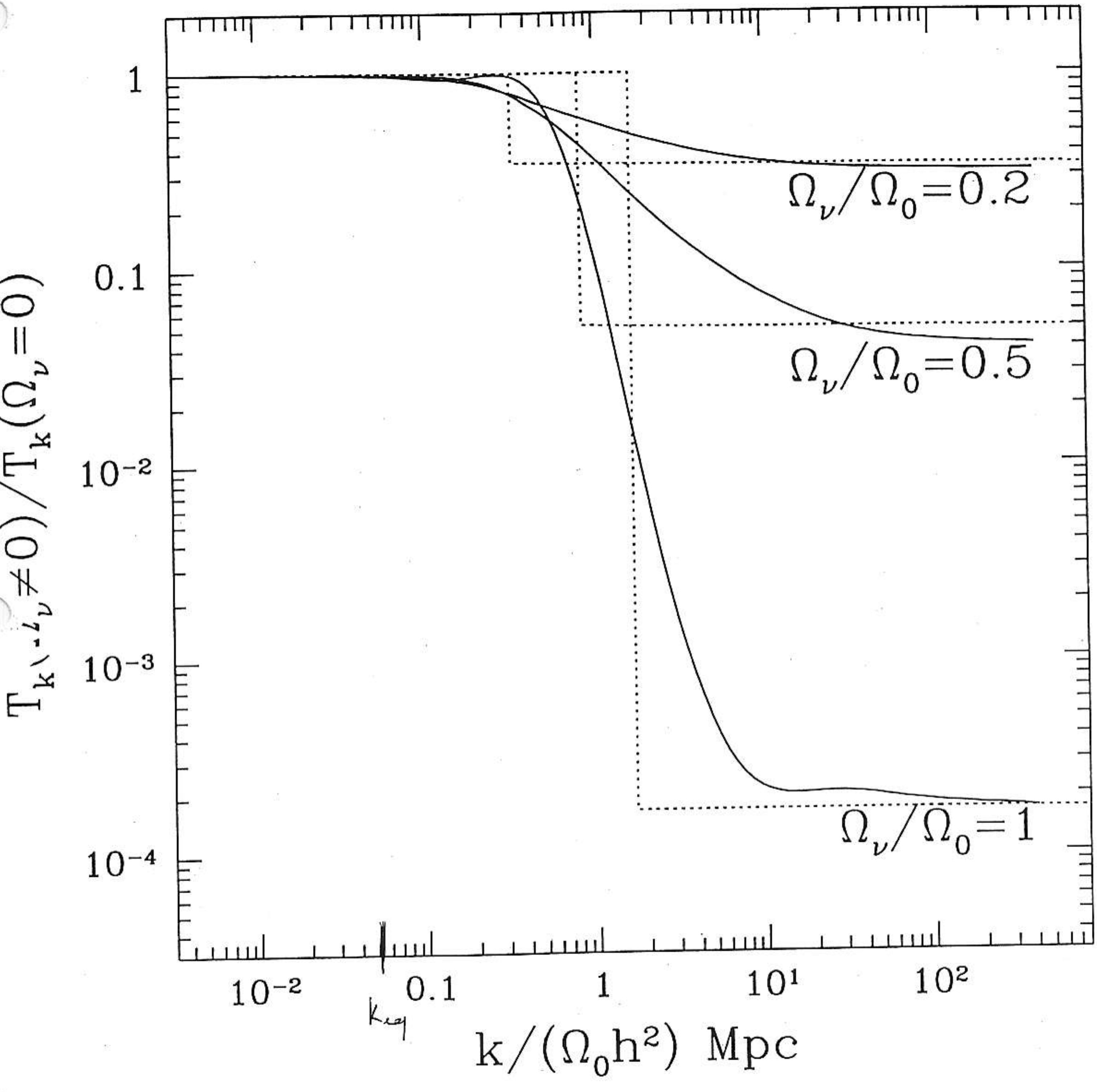
$$\left(\frac{a^\nu_{\text{non-rel}}}{a_{eq}} = 0.03 \frac{\Omega_0}{\Omega_\nu} \leq 1 \text{ if } \Omega_\nu \gtrsim 0.03\,\Omega_0 \right)$$

$$k_\nu \propto k_{eq} \frac{a_{eq}}{a^\nu_{\text{non-rel}}} = 1.6 \left(\Omega_\nu h^2\right) \mathrm{Mpc}^{-1}$$

free-streaming scale







1
0.1
10−2
10−3
10−4
10−2
0.1
1
10¹
10²
k_eq
k/(Ω₀h²) Mpc
T_k(Ω_ν≠0)/T_k(Ω_ν=0)
Ω_ν/Ω₀=0.2
Ω_ν/Ω₀=0.5
Ω_ν/Ω₀=1