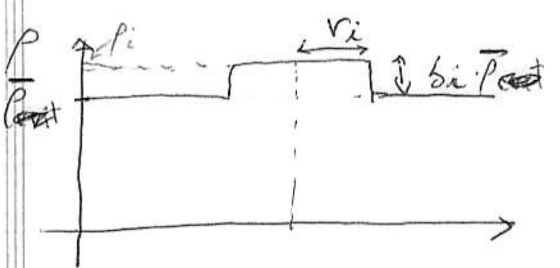
Large scale structure:

Non Linear Evolution

1) Top hat collapse:

Let's consider the non-linear evolution of the simplest perturbation: a homogeneous sphere in a matter-dominated universe.



$$\rho_i = (1+\delta_i)\,\bar{\rho} \rightarrow \frac{\Delta\rho}{\bar{\rho}} = \delta_i$$

$$\Delta\rho = \rho_i - \bar{\rho}$$

+ no shell crossing:

In the non-relativistic limit (~~Newtonian theory~~):

$$\frac{d^2 r(t)}{dt^2} = -\frac{GM}{r^2} \Rightarrow \frac{1}{2}\dot{r}^2 = \frac{GM}{r} + \text{constant}$$

$$\left(\cancel{\dot{r}}\,\ddot{r} = -\frac{GM}{r^2}\cancel{\dot{r}} \quad \text{c.v.d}\right)$$

$$M = \frac{4\pi}{3} r_i^3 \rho_i = \frac{4\pi}{3} r_i^3 \bar{\rho}_i (1+\delta_i)$$

$\rightarrow$ mean density $\bar{\rho}_i = \Omega_m \rho_{crit} a_i^{-3}$

a) Initial conditions: (assume $\Omega_m = 1$)

(1) $\dot{r}_i = H_i r_i$ (Hubble flow)

$$a = \left(\frac{3}{2} H_0 \cancel{\sqrt{\Omega_m}}\, t\right)^{\frac{2}{3}} \qquad \bar{\rho}_i = \left(\frac{3 H_0^2 \Omega_m}{8\pi G a^3}\right) = \frac{\Omega_m}{6\pi G t_i^2}$$

(matter dominated universe)

$$H = \frac{2}{3}\frac{1}{t}$$

Thus:

small ($\delta_i \ll 1$)

(1) $\dot r_i = \frac{2}{3}\frac{r_i}{t_i}$; $M = \frac{2 r_i^3(1+\delta_i)}{9Gt_i^2} = \frac{3(1+\delta_i)}{4G}\,\dot r_i^3\, t_i$

$$\frac{1}{2}\frac{4}{9}\frac{r_i^2}{t_i^2} = \frac{2G r_i^3(1+\delta_i)}{9Gt_i^2\, r_i} + \text{const}$$

$$\Downarrow$$

$$\text{const} = -\frac{1}{2}\delta_i \dot r_i^2 \qquad\qquad -\delta_i\frac{2}{9}\frac{r_i^2}{t_i^2}$$

→ behaves like a closed universe

$$\frac{1}{2}\dot r^2 = \frac{GM}{r} - \frac{1}{2}\delta_i \dot r_i^2$$

We can solve this equation parametrically:

$$r = \frac{GM}{\delta_i \dot r_i^2}(1-\cos\theta) \xrightarrow{\theta=0} r=0$$

$$t = \frac{GM}{\delta_i^{3/2}\dot r_i^3}(\theta-\sin\theta) \xrightarrow{\theta=0} t=0$$

$\frac{GM}{\delta_i \dot r_i^2} = \frac{1}{\delta_i}\frac{9t_i^2}{4r_i^2}\frac{2r_i^3(1+\delta_i)}{9Gt_i^2} = r_i\frac{(1+\delta_i)}{2\delta_i}$

$\frac{GM}{\delta_i^{3/2}\dot r_i^3} = \frac{3}{4}\frac{(1+\delta_i)}{\delta_i^{3/2}}\,t_i$

check

$$\dot r = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\Big/\frac{dt}{d\theta} = \frac{GM}{\delta_i \dot r_i^2}\sin\theta \Big/ \frac{GM}{\delta_i^{3/2}\dot r_i^3}(1-\cos\theta) =$$

$$= \frac{\sin\theta}{1-\cos\theta}\,\delta_i^{1/2}\dot r_i$$

$$\frac{1}{2}\left(\frac{\sin\theta}{1-\cos\theta}\right)^2 \delta_i \dot r_i^2 = \frac{\delta_i \dot r_i^2}{1-\cos\theta} - \frac{1}{2}\delta_i \dot r_i^2$$

$$\frac{1-\cos^2\theta}{(1-\cos\theta)^2} = \frac{\overset{1+\cos\theta}{2-1+\cos\theta}}{1-\cos\theta} = \frac{(1-\cos\theta)(1+\cos\theta)}{(1-\cos\theta)^2} \qquad \text{c.v.d.}$$

$$\left(\to \frac{\cos\theta}{\sin\theta} \to \frac{\sin\theta}{\cos\theta} \to 0 \right)$$

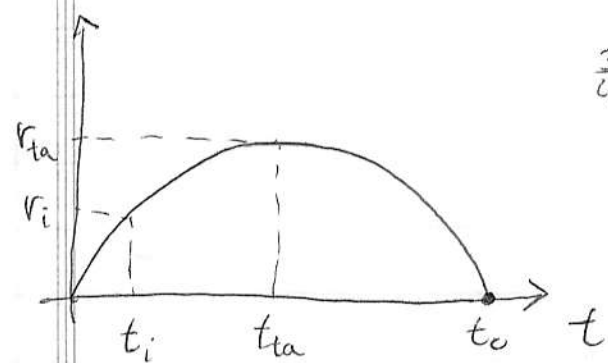
At $\theta = \pi$ $\quad \dot r \propto \frac{\sin\theta}{1-\cos\theta} \longrightarrow 0$ (turn around)

$$r_{ta} = \frac{2GM}{\delta_i \dot r_i^2} \qquad t_{ta} = \pi \frac{GM}{\delta_i^{3/2} \dot r_i^3}$$

At $\theta = 2\pi$ $\quad r \to 0$ (collapse)

$$t_c = 2 t_{ta} = 2\pi \underbrace{\frac{GM}{\delta_i^{3/2} \dot r_i^3}}_{\frac{3}{4} t_i (1+\delta_i)} = \frac{3\pi}{2} \frac{1+\overset{\nearrow 0}{\delta_i}}{\delta_i^{3/2}} t_i \simeq \frac{3\pi}{2} \frac{t_i}{\delta_i^{3/2}}$$

if $\delta_i \ll 1$



$$t_{ta} = \frac{3\pi}{4} \frac{t_i}{\delta_i^{3/2}} \Rightarrow \frac{t_{ta}}{t_i} = \left(\frac{3\pi}{4} \frac{1}{\delta_i^{3/2}} \right)$$

(2) The mean density at turn around is:

$$\rho_{ta} = \frac{3}{4\pi} \frac{M}{r_{ta}^3} = \frac{3}{4\pi} \frac{\delta_i^3 \dot r_i^6}{8 G^3 M^2} = \frac{\delta_i^3}{6\pi G t_i^2}$$

using expression for $M = \frac{3}{4G} \dot r_i^3 t_i$

$$\bar\rho (t = t_{ta}) = \frac{1}{6\pi G t_{ta}^2} \Longrightarrow \frac{\rho_{ta}}{\bar\rho_{ta}} = 1 + \delta_{ta} = \delta_i^3 \left(\frac{t_{ta}}{t_i} \right)^2 = \frac{\cancel{\delta_i^3}}{\cancel{\delta_i^3}} \left(\frac{3\pi}{4} \right)^2$$

$$1+\delta_{ta} = \frac{\rho_{ta}}{\bar\rho_{ta}} = \frac{9\pi^2}{16} \simeq 5.55$$

→ What happens at collapse?

Density should go to ∞ but in reality we have virialization due to non-sperical collapse & shell crossing:

(3) Virial theorem:

$$\underset{\text{kinetic energy}}{2K} = +\frac{GM}{r_{vir}} = \underset{\text{potential energy}}{-W} \qquad \boxed{K = -\frac{W}{2}}$$

$$E_{VIR} = K + W = \frac{W}{2} = -\frac{1}{2}\frac{GM}{r_{vir}}$$

$$E_{ta} = \underset{\substack{\| \\ 0 \\ (\dot{r}=0)}}{K} + W = W = -\frac{GM}{r_{ta}}$$

Conservation of Energy $\Rightarrow E_{vir} = E_{ta} \Rightarrow \boxed{r_{vir} = \frac{1}{2} r_{ta}}$

Average density at virialization is:

$$\rho_{vir} = \frac{4\pi}{3}\frac{M}{r_{vir}^3} = 8 \cdot \rho_{ta}$$

$$\bar{\rho}(t = t_{VIR}) = \frac{1}{4}\bar{\rho}(t = t_{ta}) \qquad t_{VIR} = t_{coll} = 2t_{ta}$$

$$\frac{\rho_{VIR}}{\bar{\rho}(t = t_{vir})} = 1 + \delta_{VIR} = 32(1 + \delta_{ta}) = 18\pi^2 = 177.7$$

- $\Delta = 177.7$ is called virial overdensity:

A virialised halo ~~has~~ mean density $\rho_{VIR}(z_{VIR}) = \Delta\,\bar{\rho}(z_{VIR})$ ~~[illegible]~~ at the redshift $z = z_{VIR}$.

- The radius of such object is called virial radius:

$$M_{DM} = \rho_{VIR}\,\frac{4\pi}{3}\,R_{VIR}^3 \longrightarrow R_{VIR} = \left(\frac{3}{4\pi}\,\frac{M_{DM}}{\rho_{VIR}}\right)^{\frac{1}{3}}$$

- The velocity dispersion inside the halo is given by virial theorem:

$$K = \frac{1}{2} v_{circ}^2 = \frac{1}{2}\,\frac{G M_{DM}}{r_{VIR}} = \frac{W}{2} \longrightarrow \underset{\substack{\uparrow \\ \text{circular velocity}}}{v_{circ}} = \sqrt{\frac{G M_{DM}}{r_{VIR}}} = \sqrt{\frac{2kT_{VIR}}{\underset{\substack{\downarrow \\ \text{if thermalised}}}{\mu}}}$$

$$\begin{cases} n_{VIR} = \dfrac{\rho_{gas}}{\mu m_{H}} = \left(0.04\ \mathrm{cm}^{-3}\right)\left(\dfrac{\Omega_b h^2}{0.019}\right)\left(\dfrac{1+z_{VIR}}{10}\right)^3 \mu^{-1} \\ R_{VIR} = (1.5\ \mathrm{kpc})\left(\dfrac{\Omega_m h^2}{0.147}\right)^{-\frac{1}{3}}\left(\dfrac{M_{DM}}{10^8 M_\odot}\right)^{\frac{1}{3}}\left(\dfrac{1+z_{VIR}}{10}\right)^{-1} \\ v_{circ} = (17\ \mathrm{km/s})\left(\dfrac{\Omega_m h^2}{0.147}\right)^{\frac{1}{6}}\left(\dfrac{M_{DM}}{10^8 M_\odot}\right)^{\frac{1}{3}}\left(\dfrac{1+z_{VIR}}{10}\right)^{\frac{1}{2}} \\ T_{VIR} = (10{,}500\ \mathrm{K})\left(\dfrac{\Omega_m h^2}{0.147}\right)^{\frac{1}{3}}\left(\dfrac{M_{DM}}{10^8 M_\odot}\right)^{\frac{2}{3}}\left(\dfrac{1+z_{VIR}}{10}\right) \end{cases}$$

~~Conclusion: non linear evolution is much faster than the linear one.~~

Specific time of collapse depends on H, δ_i, $\dot r_i$ ($\sim H_i r_i$) etc... It is convenient to use a different measure of time related to the linear overdensity:

In linear theory:

$$\begin{cases} \delta_L(t) = A\delta_i\left(\frac{t}{t_i}\right)^{\frac{2}{3}} + B\,\delta_i\,\frac{t_i}{t} \\ \dot\delta_L(t_i) = \frac{2}{3}A\left(\frac{\delta_i}{t_i}\right) - B\,\frac{\delta_i}{t_i} \end{cases}$$

with $A+B=1$
because at t_i $\delta_L = \delta_{TH}$

For top-hat: → no shell crossing

$$\rho r^3 = \rho_i r_i^3 \implies (1+\delta) = \frac{\rho}{\bar\rho(t)} = \frac{\rho_i}{\bar\rho(t)}\frac{r_i^3}{r^3}$$

Since $\begin{cases} \bar\rho \propto t^{-2} \\ \bar\rho(t) = \bar\rho_i\left(\frac{t}{t_i}\right)^{-2} \end{cases}$ $\quad (1+\delta) = \frac{r_i^3}{r^3}(1+\delta_i)\left(\frac{t}{t_i}\right)^2$

$$\dot\delta_{TH} = -3\,\frac{r_i^3}{r^3}\,\frac{\dot r}{r}\,(1+\delta_i)\left(\frac{t}{t_i}\right)^2 + 2\,\overbrace{\frac{r_i^3}{r^3}(1+\delta_i)\left(\frac{t}{t_i}\right)^2}^{(1+\delta)}\cdot\frac{1}{t} =$$

$$\Rightarrow \dot\delta_{TH} = -3(1+\delta)\frac{\dot r}{r} + 2(1+\delta)\frac{1}{t} = -3(1+\delta)\frac{\dot r}{r} + \frac{2}{t}(1+\delta)$$

$$= (1+\delta)\left(\frac{2}{t} - 3\frac{\dot r}{r}\right) = 0$$

$\left(H = \frac{2}{3}\frac{1}{t}\right)$ ⟶

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$$\dot{\delta}_{TH} = (1+\delta)\left(\frac{2}{t} - 3\frac{\dot{r}}{r}\right) \quad \text{at } t = t_i$$

$$\dot{\delta}_{TH,i} = (1+\delta_i)\left(\frac{2}{t_i} - 3H_i\right) = 0 \qquad H_i = \frac{2}{3}\frac{t}{t_i}$$

So we want $\dot{\delta}_{L,i} = 0 \Rightarrow \frac{2}{3}A = B$

$A + B = 1 \Rightarrow A = \frac{3}{5}$, $B = \frac{2}{5}$

$$\delta_L = \frac{3}{5}\delta_i\left(\frac{t}{t_i}\right)^{\frac{2}{3}} + \frac{2}{5}\delta_i\left(\frac{t_i}{t}\right) \xrightarrow{t \gg t_i} \frac{3}{5}\delta_i\left(\frac{t}{t_i}\right)^{\frac{2}{3}}$$

At late times:

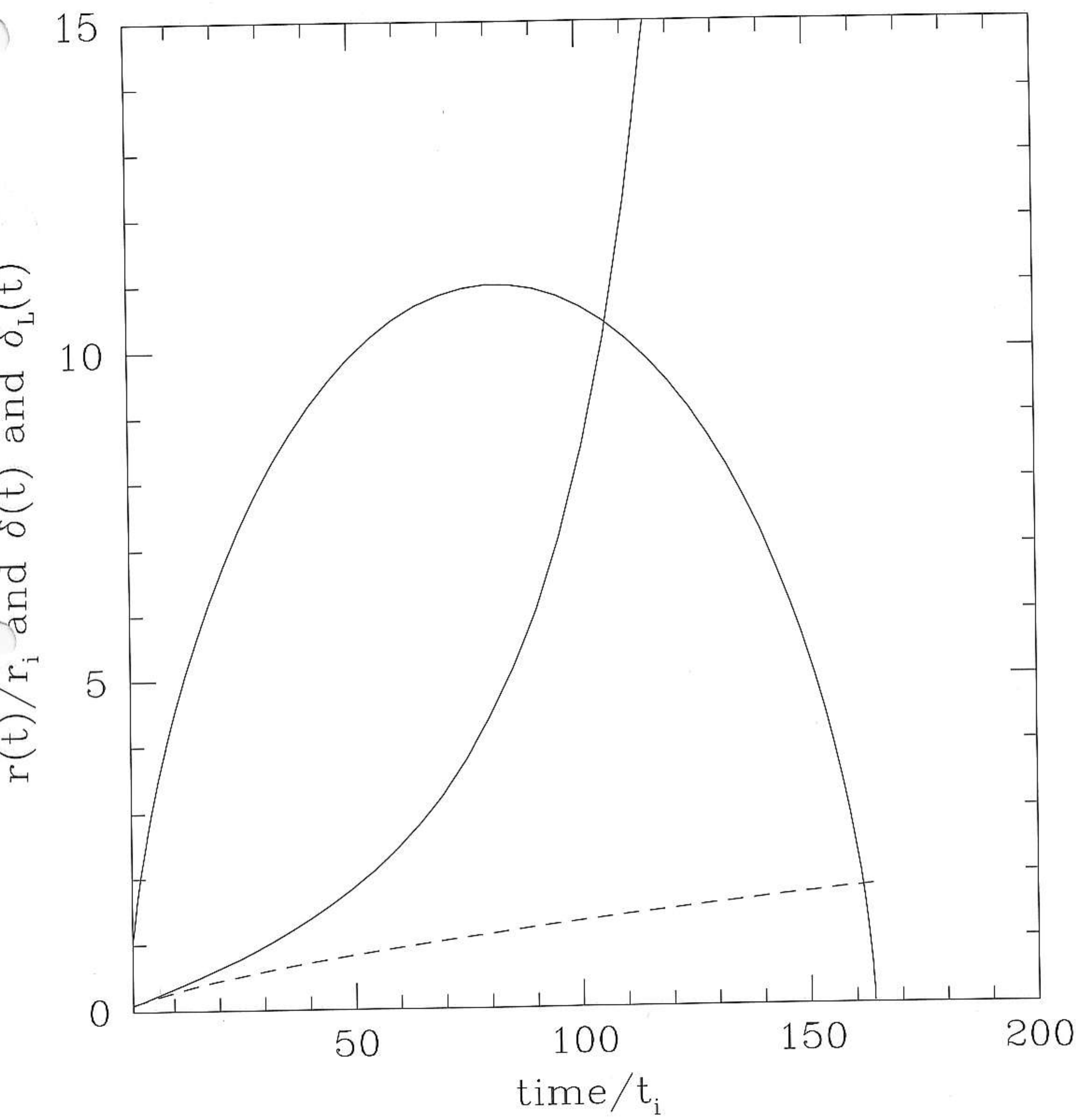
At collapse:

$$\delta_{L,coll} = \frac{3}{5}\delta_i\left(\frac{2t_{ta}}{t_i}\right)^{\frac{2}{3}} = \frac{3}{5}2^{\frac{2}{3}}\underbrace{(1+\delta_{ta})}_{\left(\frac{3\pi}{4}\right)^2}^{\frac{1}{3}} = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{\frac{2}{3}} \simeq 1.68$$

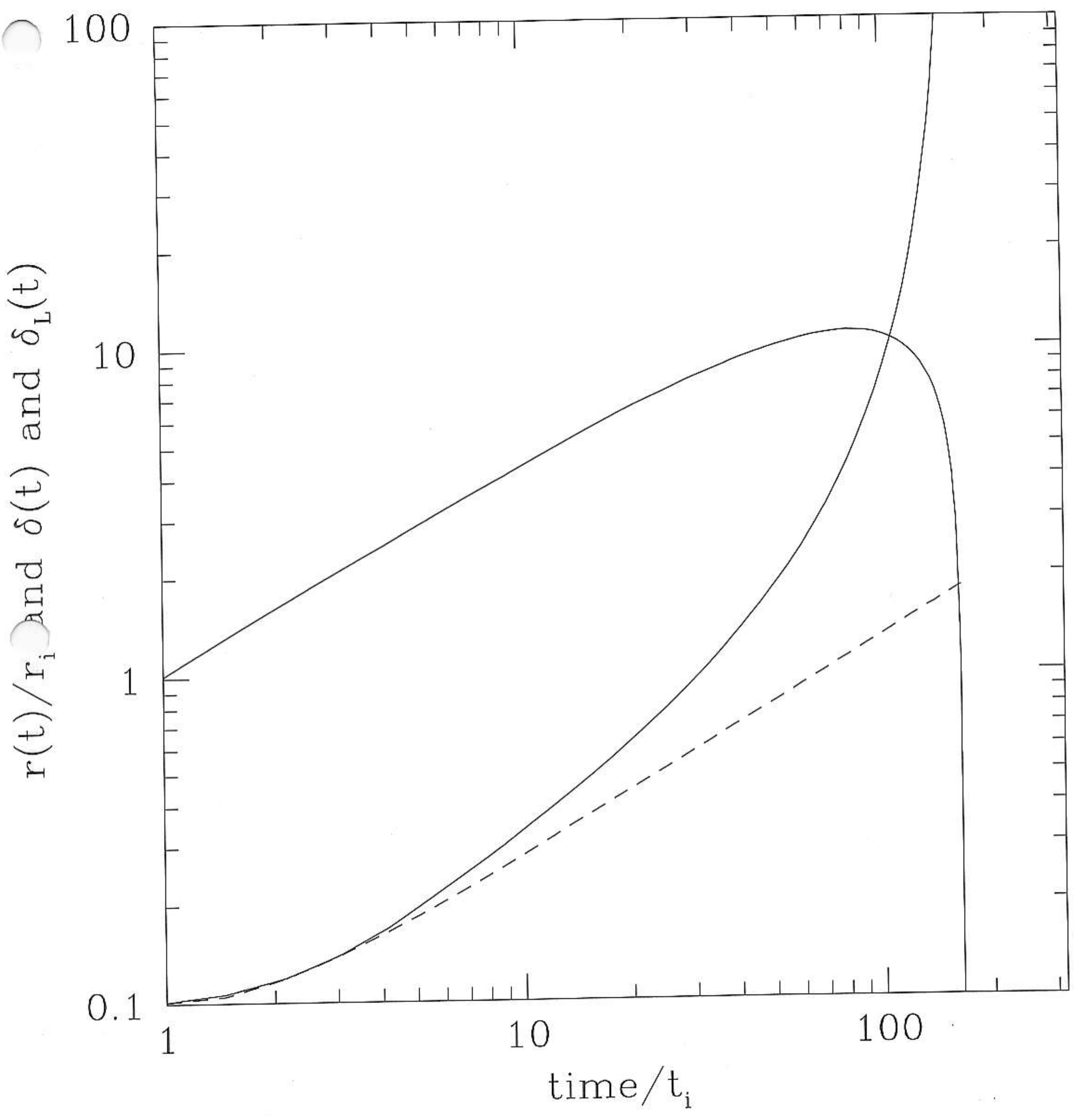
$$\left[\; 1+\delta_{ta} = \delta_i^3\left(\frac{t_{ta}}{t_i}\right)^2 \;\right]$$

This demonstrates also that $\delta_L \sim 1$ corresponds to non-linear regime

Conclusion: non linear evolution is much faster than the linear one



15
10
5
0
50
100
150
200
time/t_i
$r(t)/r_i$ and $\delta(t)$ and $\delta_L(t)$



100
10
1
0.1
1
10
100
$r(t)/r_i$ and $\delta(t)$ and $\delta_L(t)$
time/t_i

Modelling the large-Scale Structure

The evolution of the collisionless dark matter in the expanding universe is described by "Vlasov" equation:
↳ or (collisionless Boltzmann eq.) ↗ $f(\vec{x},\vec{v};t)$

$$\frac{\partial f}{\partial t} + \frac{\vec{v}}{a}\frac{\partial f}{\partial \vec{x}} - \Big(\underbrace{\frac{1}{a}\frac{\partial \varphi}{\partial \vec{x}}}_{g_{grav}} + \underbrace{H\vec{v}}_{\text{Hub. expansion}}\Big)\frac{\partial f}{\partial \vec{v}} = 0$$

In practice, this is a 6-dimensional problem that can be solved with method of characteristics
(see back of page)

comoving distance

peculiar velocity

$$\begin{cases} \dfrac{d\vec{x}}{dt} = \dfrac{\vec{v}}{a} \quad \text{(peculiar velocity)} \\[2ex] \dfrac{d\vec{v}}{dt} = -H\vec{v} - \dfrac{1}{a}\dfrac{\partial \varphi}{\partial \vec{x}} \end{cases} \qquad \text{or} \qquad \begin{cases} \vec{v}_{pec} = a\dfrac{d\vec{x}}{dt} \\[2ex] a\left(\dfrac{d\vec{v}}{dt} + \dfrac{1}{a}\dfrac{da}{dt}\vec{v}\right) = -\dfrac{\partial \varphi}{\partial \vec{x}} \end{cases}$$

$$\frac{d\vec{v}}{dt} + H\vec{v} = -\frac{\partial \varphi}{\partial \vec{r}_{ph}} \qquad\qquad a\frac{d\vec{v}}{dt} + \frac{da}{dt}\vec{v} = \frac{d(a\vec{v})}{dt} = -\frac{\partial \varphi}{\partial \vec{x}}$$

N.B.: $\vec{r}_{ph} = a\cdot\vec{x} \Rightarrow \vec{v}_{ph} = \dfrac{d\vec{r}_{ph}}{dt} = \dfrac{da}{dt}\vec{x} + a\dfrac{d\vec{x}}{dt} = a\left(H\vec{x} + \dfrac{d\vec{x}}{dt}\right) = H\vec{r}_{ph} + a\dfrac{d\vec{x}}{dt}$

$$\boxed{\vec{v}_{ph} = \underset{H\vec{r}_{ph}}{\underset{\parallel}{\vec{v}_{Hubble}}} + \underset{a\frac{d\vec{x}}{dt}}{\underset{\parallel}{\vec{v}_{pec}}}}$$

Numerical integration for N-particles $i = 1, 2, \dots N$

$$\vec{x}_i(t+\Delta t) = \vec{x}_i(t) + \Delta t \frac{1}{a(t)}\vec{v}_i(t)$$

$$\vec{v}_i(t+\Delta t) = \vec{v}_i(t) + \Delta t\left[H(t)\vec{v}_i(t) + \frac{1}{a(t)}\vec{F}_i(t)\right]$$

where the ~~force~~ gravitational force is:

$$\vec{F}_i(t) = \frac{\partial \varphi}{\partial \vec{x}}\left[\vec{x} = \vec{x}_i(t)\right]$$

There are many algorithms to find $\vec{F}_i(t)$:

1) Direct summation: PP = particle-particle

$$\vec{F}_i = \sum_{j=1;\, j\neq i}^{N} \frac{G m_j \vec{r}_{ij}}{r_{ij}^2}$$

accurate but very slow: (time $\propto N^2$)

special hardware: <u>GRAPE</u>

2) PM = particle-mesh:

~~finding~~ the force calculating the density field ~~on~~ a mesh. Very fast but limited spatial resolution. (time $\propto N \log N$)

3) P^3M = particle-particle-particle-mesh:

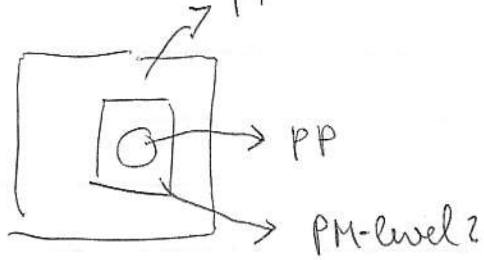
a combination of PP + PM

time $\propto N \log N + N_{pp}^2$

N_{pp} = number of particles in PP region.

4) $\underline{AP^3M}$ = Adaptive P^3M

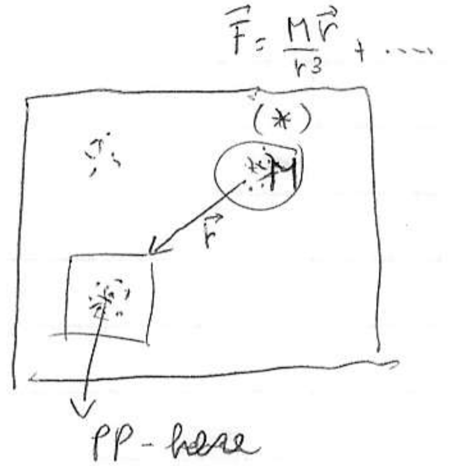
faster than P^3M



$\text{time} = N_1 \log N_1 + N_2 \log N_2 + N_{PP}^2$

5) <u>Tree codes:</u>

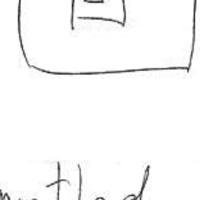
Very popular in cosmology: efficient when particles are highly clustered



(*) Treated as a point particle + dipole term + quadrupole + ...

6) AMR = adaptive mesh refinement

Similar to AP^3M but without PP part.



$\text{time} \propto N_1 \lg N_1 + N_2 \log N_2 + \dots$

7) ~~[illegible]~~ SCF = self consistent Field method

$$\varphi(\vec{x}) = \sum_{j = \text{multipoles}} A_j \, \varphi_j(\vec{x})$$

→ e.g. spherical harmonics in spherical geometry.

Initial Conditions:

Initial conditions are given by linear theory:

$$\rho = \bar{\rho}\,(1+\delta) \qquad |\delta| << 1$$

From Vlasov equation:

1) $$\frac{d\vec{v}_s}{dt} = \frac{\partial \vec{v}_s}{\partial t} + \frac{1}{a}\,\vec{v}_s \cdot \vec{\nabla}\,\vec{v}_s = -\cancel{2}H\vec{v}_s - \frac{1}{a}\frac{\partial \varphi}{\partial \vec{x}}$$

2) $$\frac{\partial \rho}{\partial t} + 3H\rho = -\frac{1}{a}\,\operatorname{div}(\rho\,\vec{v}_s) \quad \leftarrow \text{mass conservation} \quad \left[\frac{D\rho}{Dt} = 0\right]$$

v_s = streaming velocity

In linear theory (1st order)

$$\begin{cases} \dfrac{\partial \delta}{\partial t} \cancel{+3H\delta} = -\dfrac{1}{a}\,\operatorname{div}\vec{v}_s \\[2ex] \dfrac{\partial \vec{v}_s}{\partial t} = -H\vec{v}_s - \dfrac{1}{a}\dfrac{\partial \varphi}{\partial x} \end{cases}$$

$$+ \left(\underset{\nabla^2}{\Delta}\varphi = 4\pi G\,\bar{\rho}\,a^2\,\delta \quad \text{Poisson Eq.} \right)$$

Going into Fourier space:

$$\begin{cases} \delta(\vec{x};t) = \dfrac{1}{(2\pi)^3}\displaystyle\int d^3k\;\delta_{\vec{k}}(t)\,e^{-i\bar{k}\bar{x}} \cancel{= \frac{1}{(2\pi)^3}\int d^3k\,\rho(k)\,e^{i\bar{k}\bar{x}}} \\[2ex] \vec{v}_s(\vec{x};t) = \dfrac{1}{(2\pi)^3}\displaystyle\int d^3k\;\delta_{\vec{k}}(t)\,e^{-i\bar{k}\bar{x}} \\[2ex] \varphi(\vec{x};t) = \dfrac{1}{(2\pi)^3}\displaystyle\int d^3k\;\varphi_{\vec{k}}(t)\,e^{i\bar{k}\bar{x}} \end{cases}$$

$$\begin{cases} \dot{\delta}_{\vec{k}} = \frac{1}{a} i\vec{k}\vec{v}_{\vec{k}} & (1) \\ \dot{\vec{v}}_{k} = -H\vec{v}_{\vec{k}} + \frac{1}{a} i\vec{k}\,\varphi_{\vec{k}} & (2) \\ -k^2\varphi_{\vec{k}} = 4\pi G\bar{\rho}\,a^2\,\delta_{\vec{k}} & (3) \end{cases}$$

$\Downarrow$

differentiate (1) + combining equations: $\ddot{\delta}_{\vec{k}} = -H\dot{\delta}_{\vec{k}} + \frac{i\vec{k}}{a}\underbrace{\left[-H\vec{v}_{k} + \frac{1}{a} ik\,\varphi_{\vec{k}}\right]}_{\dot{v}_{\vec{k}}} \Rightarrow$

$$\ddot{\delta}_{\vec{k}} = -2H\dot{\delta}_{\vec{k}} + 4\pi G\bar{\rho}\,\delta_{\vec{k}}$$

same as previously derived for Jeans instability $\Rightarrow$ $\boxed{\ddot{\delta}_{\vec{k}} + 2H\dot{\delta}_{k} = 4\pi G\bar{\rho}\,\delta_{\vec{k}}}$

$\Downarrow$

$$|\delta_{\vec{k}}| \simeq D_+(t)$$

Thus: $\delta_{\vec{k}} = D_+(t)\,A_{\vec{k}}$ (4)

$\hookrightarrow$ random vectors (Gaussian distribution)

$|\delta_{\vec{k}}|^2 \propto P(k)$ = power spectrum

$$\delta_{\vec{k}} = \frac{D_+(t)}{D_+(t_0)}\sqrt{P(k,t_0)}\cdot\lambda_{\vec{k}}$$

$\uparrow$ random unit vectors ($|\lambda_{\vec{k}}|^2 = 1$)

From (4) differentiating $\Rightarrow$ $\dot{\delta}_{k} = \frac{\dot{D}_+}{D_+}\delta_{\vec{k}} = \frac{1}{a} i\vec{k}\vec{v}_{\vec{k}}$

$i\vec{k}\,\frac{\dot{D}_+}{D_+}\,\delta_{\vec{k}}\,a = k^2\vec{v}_{\vec{k}} \Rightarrow \boxed{\vec{v}_{\vec{k}} = -i\vec{k}\,\dot{a}\,f\,\frac{\delta_{\vec{k}}}{k^2}}$ where

$$f = \frac{a}{\dot{a}} \frac{\dot{D}_+}{D_+} = \frac{1}{H} \frac{\dot{D}_+}{D_+} = f(\Omega_m) \simeq \Omega_m^{0.6}$$

- In practice IC are generated using the following steps:

1) Start from uniform grid of point particles.

2) Generate random vectors $\lambda_{\vec{k}}$

3) Multiply $\lambda_{\vec{k}}$ by $\left(\sqrt{P(k,t_0)}\,\frac{D_+(t_i)}{D_+(t_0)}\right)$ to find $\delta_{\vec{k}}$ where t_i = starting time of the simulation

4) Find $\vec{v}_k \propto \delta_{\vec{k}}$

5) Inverse FFT to get $\vec{v}(\vec{x};t_i)$.

6) Derive initial displacement of uniform grid points $\Delta x \simeq \vec{v}(x;t_i)\cdot t_i$

$\Downarrow$

Start N-body calculation

PLATE 2

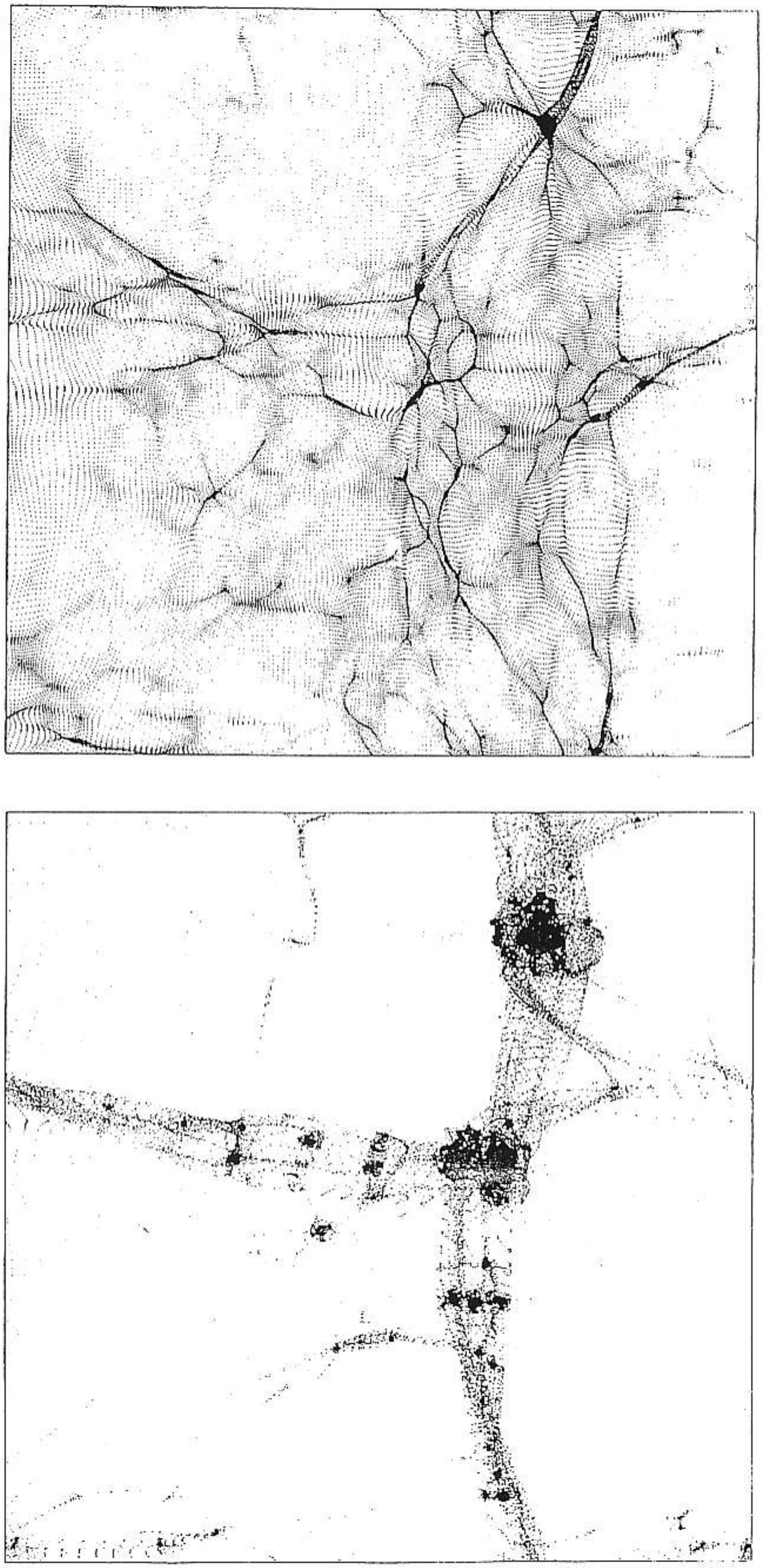


FIG. 5.—(*a*) is N32F3. (*b*) N32F5.

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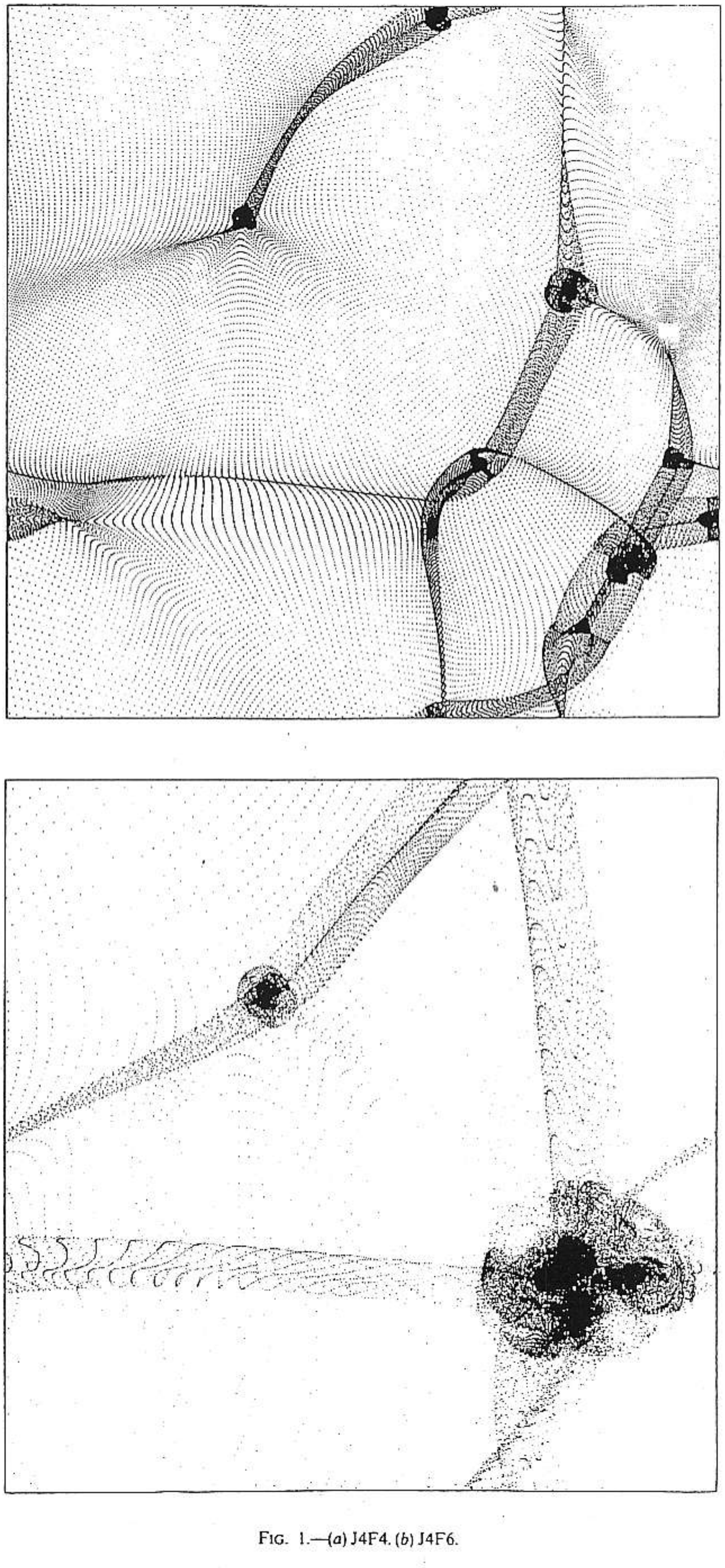


FIG. 1.—(a) J4F4. (b) J4F6.

Variance of density Perturbations:

$$\sigma^2(R,t_0) = \frac{1}{2\pi^2}\int_0^\infty k^2\, dk\; P(k,t_0)\; W^2(k\cdot R)$$

variance — window function

- Let's see why we need a window function:

$$\delta(x,t_i) = \frac{1}{(2\pi)^3}\,\frac{D_+(t_i)}{D_+(t_0)}\int \underbrace{d^3k}_{(k^2\,dk\,d\Omega)}\; e^{-i\vec{k}\vec{x}}\sqrt{P(k,t_0)}\,\lambda_{\vec{k}}$$

$$\Downarrow$$

$$\sigma^2 \equiv \langle|\delta|^2\rangle = \frac{1}{V}\int \delta^2\, dV = \left(\frac{D_+(t_i)}{D_+(t_0)}\right)^2 \frac{1}{(2\pi)^3}\cdot 4\pi \int_0^\infty k^2\, dk\; P(k)$$

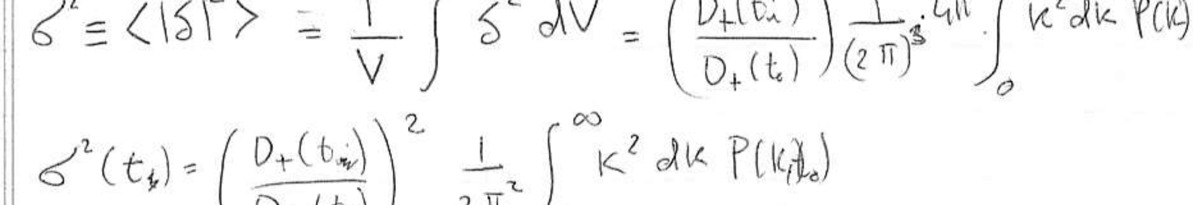
$$\sigma^2(t_i) = \left(\frac{D_+(t_i)}{D_+(t_0)}\right)^2 \frac{1}{2\pi^2}\int_0^\infty k^2\, dk\; P(k,t_0)$$

assuming $P(k)\propto k^n$

$$\propto \int_0^\infty k^{2+n}\, dk \propto k^{3+n}\Big|_0^\infty \xrightarrow[k\to\infty]{} \infty$$

$\boxed{\text{if } n > -3}$ diverges

$\boxed{\sigma^2 \xrightarrow[k\to\infty]{} \infty}$ for $n>-3$



δ diverges on small scales for spectrum of pert. with $n>-3$

We have to use a filter to smooth the small scale density perturbations.

$$\delta_R(\vec{x},t) = \underbrace{\frac{1}{V}}_{\propto R^3} \int dV' \, \delta(\vec{x}',t) \, w\left(\frac{|\vec{x}-\vec{x}'|}{R}\right) \qquad \text{(convolution)}$$

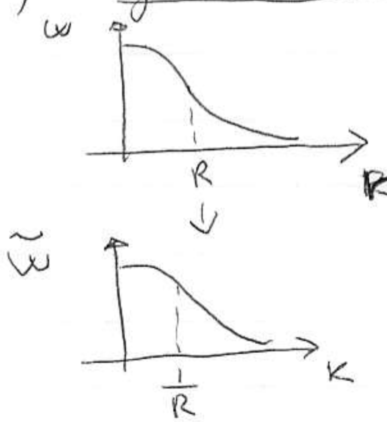
$$\Downarrow$$

$$\sigma_R^2 = \langle |\delta_R|^2 \rangle = \frac{1}{2\pi^2} \left(\frac{D_+(t_i)}{D_+(t_0)}\right)^2 \int_0^\infty k^2 dk \, P(k) \, \tilde{W}(kR)$$

$\tilde{W}(kR) \to$ Fourier transform of w.

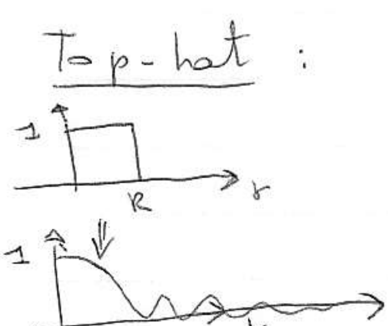
If $\tilde{W}(kR) \to 0$ for $k \to \infty$ (small scales) $\Rightarrow \sigma_R$ does not diverge

a) Gaussian window function.



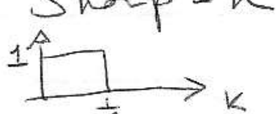
$$\begin{cases} w\left(\frac{r}{R}\right) = \frac{1}{(2\pi R^2)^{\frac{3}{2}}} \exp\left[-\frac{r^2}{2R^2}\right] \\ \tilde{W}(kR) = e^{-\frac{k^2R^2}{2}} \end{cases}$$

b) Top-hat :



$$\begin{cases} w\left(\frac{r}{R}\right) = \frac{3}{4\pi R^3} \begin{cases} 1 & r<R \\ 0 & r>R \end{cases} \\ \tilde{W}(kR) = 3\,\frac{\sin(kR) - kR\cos(kR)}{(kR)^3} \end{cases}$$

c) Sharp-k



$$\tilde{W} = \begin{cases} 1 & k<\frac{1}{R} \\ 0 & k>\frac{1}{R} \end{cases}$$

Let's use for simplicity a sharp-k filter and $P(k) \propto k^n$

$$\sigma_R^2 \propto \int_0^{\frac{1}{R}} P(k)\, k^2\, dk = \int_0^{\frac{1}{R}} k^{n+2}\, dk \propto \left(\frac{1}{R}\right)^{n+3}$$

$$\sigma_R \propto R^{-\frac{n+3}{2}}$$

$$M \propto R^3 \rightarrow R \propto M^{\frac{1}{3}}$$

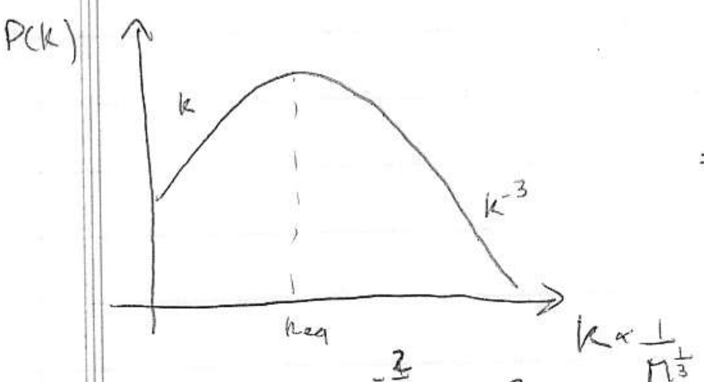
$$\boxed{\sigma_R \propto M^{-\frac{n+3}{6}}} \Rightarrow \begin{cases} n<-3 & \sigma_R \propto M^{\alpha} \ (\alpha>0) \text{ increases with increasing } M \\ n=-3 & \sigma_R = \text{const} \\ n>-3 & \sigma_R \propto \frac{1}{M^{\alpha}} \ (\alpha>0) \text{ decreases with increasing } M \end{cases}$$

⇒ Important conclusion:

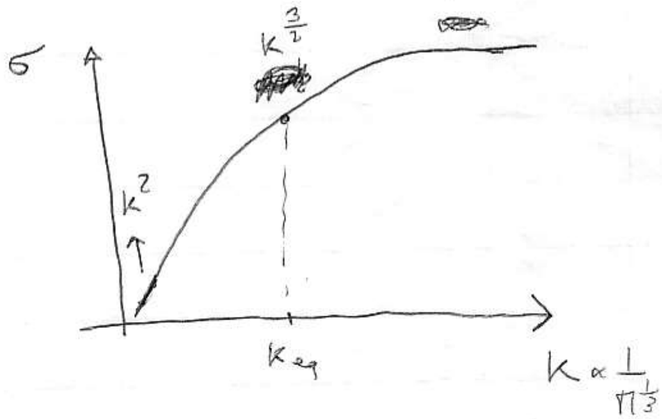
$n<-3$ top-down clustering (eg. Hot dark matter)

$n>-3$ bottom-up clustering (eg. Cold-dark Matter)

CDM



$\Rightarrow$



$n=1 \quad \sigma_R \propto M^{-\frac{2}{3}} \propto k^2$

$n=0 \quad \sigma \propto M^{-\frac{1}{2}} \propto k^{\frac{3}{2}}$

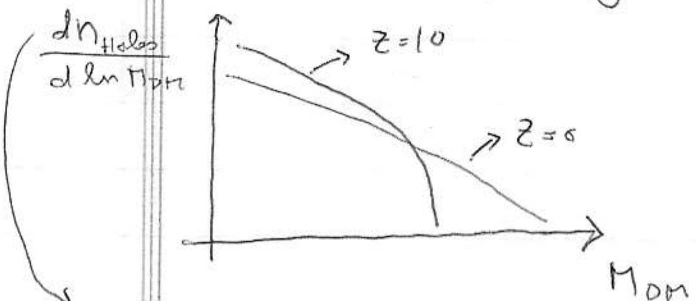
$n=-1 \quad \sigma \propto M^{-\frac{1}{3}} \propto k$

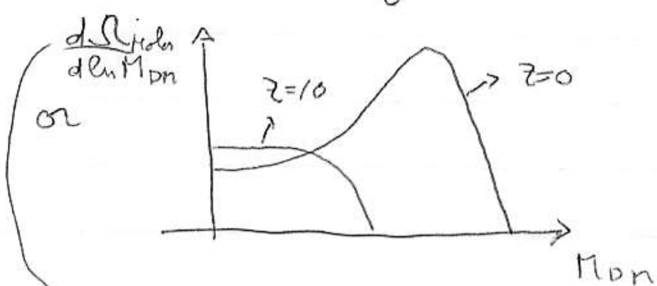
$n=-2 \quad \sigma \propto M^{-\frac{1}{6}} \propto k^{\frac{1}{2}}$

$n=-3 \quad \sigma \propto \text{const} \propto \text{const}$

Press-Schechter Formalism:

This is a very powerfull formula to ~~...~~ estimate the number density of ~~galaxies~~ DM halos of a given mass at any redshift.





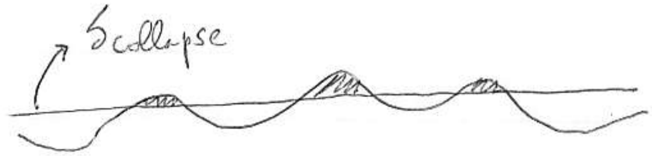
$n_{Halos} \equiv$ number density $[Mpc^{-3}]$

$\Omega_{Halos} \equiv \frac{\rho_{Halos}}{\rho_{DM}}$ = fraction of the DM in halos of mass M_{DM}

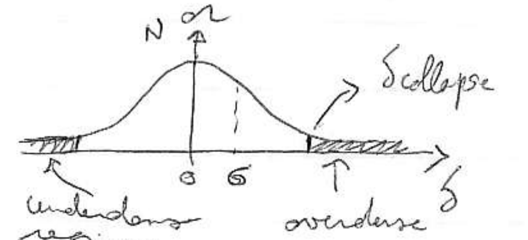
NOTE : $1 = \int \frac{d\Omega}{d\ln M_{DM}} \cdot d\ln M_{DM}$

Let's consider a gaussian random field of density perturbations:

$$P(\delta) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{\delta^2}{2\sigma^2}}$$



The fraction of the ~~volume~~ mass that collapses is:



$$\Omega_c = \int_{\delta_c}^{\infty} P(\delta)\, d\delta = \frac{1}{2}\left[1 - \mathrm{erf}\left(\frac{\delta_c}{\sqrt{2}\,\sigma}\right)\right]$$

consider only overdense regions

So if $\frac{\delta}{\sigma} \to \infty$ $\quad \Omega_c \to \frac{1}{2}$ only half of the universe collapses. This is not realistic because else underdense regions collapse when they find themself in a larger collapsing volume.

Press & Schechter in 1974 simply multiplied $\Omega_c \times 2$ to account for underdense regions.
(NOTE: this formula agrees very well with results of N-body simulations!)

$$\Omega_c(>M) \; 1 - \mathrm{erf}\left(\frac{\delta_c}{\sqrt{2}\,\sigma(M)}\right)$$

<u>NOTE</u>: error function $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$

variance on scale M or R where $M = \frac{4\pi}{3}\bar{\rho} R^3$

<u>N.B.</u>: More precisely the definition of M depends on the "filter function" used to calculate $\sigma(M)$.

From Top-hat collapse model can be shown that $\boxed{\delta_c \simeq 1.69}$ is the overdensity treshold to have non-linear collapse.

$$\cancel{\frac{d\Omega_c}{dM} = -\frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{\delta_c^2}{2\sigma^2}} \left(-\frac{\delta_c}{\sqrt{2}\,\sigma^2}\right) \frac{d\sigma}{dM}}$$

$$\boxed{\frac{d\Omega_c}{dM} = -\frac{d\,\mathrm{erf}(x)}{dx}\frac{dx}{dM} = \sqrt{\frac{2}{\pi}}\,\frac{\delta_c}{\sigma^2}\frac{d\sigma}{dM}\, e^{-\frac{\delta_c^2}{2\sigma^2}}}$$

$$\left\{ \begin{aligned} x &= \frac{\delta_c}{\sqrt{2}\,\sigma} \\ \frac{d\,\mathrm{erf}(x)}{dx} &= \frac{2}{\sqrt{\pi}}\, e^{-x^2} \\ \frac{dx}{dM} &= -\frac{\delta_c}{\sqrt{2}\,\sigma^2}\frac{d\sigma}{dM} \end{aligned} \right.$$

The # density of holes is: $\frac{dn}{dM} = \frac{1}{M} \frac{d\bar{\rho}\Omega_c(<M)}{dM}$

$(M \cdot dn = \bar{\rho}\, d\Omega_c)$

$$\boxed{\frac{dn}{dM} = \bar{\rho}\sqrt{\frac{2}{\pi}}\, \frac{\delta_c}{M\sigma^2(M)} \frac{d\sigma}{dM}\, e^{-\frac{\delta_c^2}{2\sigma^2}}}$$

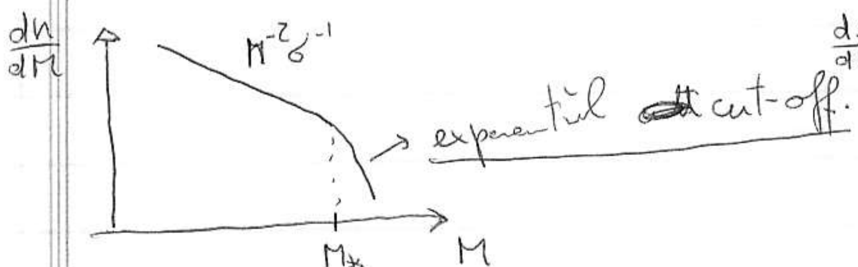
$\sigma(M)$ decreases with increasing M: <u>in CDM</u>

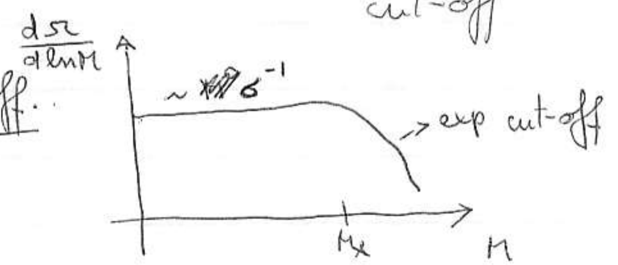
$\begin{cases} \text{large } M \rightarrow \text{small } \sigma \\ \text{small } M \rightarrow \text{large } \sigma \end{cases}$

small M $\rightarrow$ $\sigma(M) \gg \delta_c \rightarrow \frac{dn}{dM} \propto \frac{1}{M\sigma^2} \frac{d\sigma}{dM}$

$\rightarrow \frac{dn}{dM} \propto \frac{1}{M^2\sigma} \frac{d\ln\sigma}{d\ln M} \simeq \frac{1}{M^2\sigma}$

large M $\rightarrow$ $\sigma(M) \ll \delta_c \rightarrow \frac{dn}{dM} \propto e^{-\frac{\delta_c^2}{2\sigma^2}}$ exponential cut-off





$\left(\frac{d\Omega}{d\ln M} \propto M^2 \cdot \frac{dn}{dM}\right)$

We found that if $P(k) \propto k^n \Rightarrow \sigma(M) \propto M^{-\frac{3+n}{6}}$

$\Rightarrow \frac{dn}{dM} \propto M^{\frac{n-9}{6}}$ $\quad \frac{d\Omega}{d\ln M} \propto M^{\frac{n+3}{6}} = \begin{cases} \text{const} & \text{Dwarfs } (n=-3) \\ M^{\frac{1}{3}} & \text{MW galaxies } (n=-1) \\ M^{\frac{1}{2}} & \text{Clusters } (n=0) \end{cases}$