ASTR 601 - Radiative Processes

HOMEWORK #4 (Tue Oct. 13, 2009)
due: Tue Oct. 20 in class

1 Fermi-Dirac and Bose-Einstein distributions (15pts)

In class we have calculated the mean particle number $\langle N \rangle$ and the pressure for a gas of photons (Bosons with chemical potential $\mu = 0$).

(a) Derive expressions for $\langle N \rangle$ and the pressure, P, for a gas of fermions in the relativistic and non-relativistic limits. Start from the expressions derived in class for the grand potential and grand canonical partition function for Fermions. The general expression for $\langle N \rangle$ and the pressure P can be written in terms of the Fermi integrals:

$$f_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1} dx}{z^{-1} \exp(x) + 1},$$

where $z = \exp(\mu/kT)$ and Γ is the Gamma function.

(b) Write down the general expressions (for $\mu \neq 0$) for $\langle N \rangle$ and P for a Boson gas in the relativistic and non-relativistic limits. You can write the results in terms of the Bose integrals:

$$g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1} dx}{z^{-1} \exp(x) - 1}.$$

(c) Derive the equation of state, $P(n, T)$, for Fermions and Bosons in the relativistic and non-relativistic limits. Explore the differences between degenerate ($\mu > kT$) and non-degenerate ($\mu \ll kT$) non-relativistic gas and degenerate ($\mu > mc^2$) relativistic gas. *Note: this problem is long but quite easy. Please, be concise*

2 Three-level quantum System (15pts)

Consider a system of two particles which can be in any of three quantum states of energy 0, Δ or 3Δ. The system is in contact with a heat reservoir at temperature T. Write down the partition function $Z(T)$ and the grand partition function $Q(\mu, T)$ for three cases:

(a) Maxwell-Boltzmann (distinguishable) particles.

(b) Fermi-Dirac (indistinguishable) particles (Fermions).

(c) Bose-Einstein (indistinguishable) particles (Bosons).

Be sure to account for the multiplicity of ways (g_i) of obtaining a given total energy $E_i = \sum n_i \epsilon_i$, where n_i is the occupation number and ϵ_i is the energy level. *Hint: For Maxwell-Boltzmann (distinguishable) particles the partition function for N particles is $Z_N = (Z_1)^N$, where Z_1 is the partition function for one particle (you can verify that). The grand partition function is related to Z_N by: $Q = \sum_{N=0}^{\infty} \exp[N\mu/kT] Z_N$.*