Plasma Effects

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Wave propagation in plasma

$$\nabla \cdot \mathbf{E} = 4\pi\rho_e \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday law of induction)}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \frac{4\pi \mathbf{J_e}}{c} + \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Amper law)}$$

Look for solution of plane monochromatic waves in the form

$$\mathbf{E} = \mathbf{e_0} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$
$$\mathbf{B} = \mathbf{b_0} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$

and suppose the plasma response is also

$$\rho_e = \rho_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$
$$\mathbf{J_e} = \mathbf{j_0} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)},$$

Maxwell's equations become:

$$\begin{aligned} i\mathbf{k}\cdot\mathbf{E} &= 4\pi\rho_e \quad i\mathbf{k}\times\mathbf{E} = i\frac{\omega}{c}\mathbf{B} \\ i\mathbf{k}\cdot\mathbf{B} &= 0 \qquad i\mathbf{k}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J_e} - i\frac{\omega}{c}\mathbf{E} \end{aligned}$$

Model of a *cold electron plasma:*

Assume electron's motion is thermal and NR: in the equation of motion of the electron we can drop $\mathbf{k}\cdot\mathbf{x} \sim x/\lambda$ term in comparison with $\omega t \sim 1$ term because during one oscillation period $x \ll \lambda$: $x = v/\nu \ll c/\nu = \lambda$. NR equation of motion of electrons is:

$$m_e\dot{\mathbf{v}} = -e\left(\mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B_0}\right)$$

For NR electrons B term in EM field drops. But we retain net $\mathbf{B_0} = B_0\hat{\mathbf{z}}$ field. For simplicity let's consider electromagnetic radiation propagating along $\mathbf{B_0}$. We will see that any component of $\mathbf{B_0}$ perpendicular to $\hat{\mathbf{k}}$ has negligible effect on the electron's motion.

We can express an arbitrarily polarized EM wave propagating in the z-direction as the sum of left- and right- circularly polarized waves:

$$\mathbf{E}_{\pm} \equiv (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})\mathcal{E}_0 e^{-i\omega t}$$

We can then search for a solution for $\mathbf{v}$ in the same form:

$$\mathbf{v} = (\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})v_0 e^{-i\omega t}$$

Substitution in equation:

$$-i\omega m_e \mathbf{v} = -e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B_0}\right)$$

gives

$$-i\omega m_e v_0 = -e\mathcal{E}_0 \mp ie\frac{v_0}{c}B_0$$

Thus,

$$\mathbf{v} = \frac{-ie\mathbf{E}}{m_e(\omega \mp \omega_L)}, \text{ where } \omega_L \equiv \frac{eB_0}{m_e c}$$

Therefore the current is,

$$\mathbf{J_e} = -n_e e \mathbf{v} = \sigma_{cond}\mathbf{E}.$$

The electric conductivity is

$$\sigma_{cond} = \frac{i n_e e^2}{m_e(\omega \mp \omega_L)} = \frac{i\omega_{pe}}{4\pi(\omega \mp \omega_L)},$$

where we have defined the plasma frequency as

$$\omega_{pe} \equiv \frac{4\pi n_e e^2}{m_e} = 5.6 \times 10^4 n_e^{1/2} \text{ rad/s}$$

if we express n_e in units of cm^{-3}. The equation above represents Ohm's law for this problem. However, the system suffers no true dissipation (for $\omega \neq \omega_L$). The electric conductivity is purely imaginary thus, as expected, we obtain

$$\langle \mathbf{J_e} \cdot \mathbf{E} \rangle = \frac{1}{4}(\mathbf{J_e^*} \cdot \mathbf{E} + \mathbf{J_e} \cdot \mathbf{E}^*) = 0.$$

The charge density can be derived from equation. of charge conservation:

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J_e} = -i\omega\rho_e + i\mathbf{k} \cdot \mathbf{J_e} = 0 \rightarrow \rho_e = \frac{\mathbf{k} \cdot \mathbf{J_e}}{\omega}$$

Now we can plug ρ_e in Coulomb's law:

$$\mathbf{k} \cdot \mathbf{E} \left(1 - \frac{\omega_{pe}^2}{\omega(\omega \mp \omega_L)} \right) = 0$$

If we identify the expression in parenthesis as the *plasma dielectric constant*, ϵ, we have $\nabla \cdot \mathbf{D} = 0$, where the displacement vector is defined as $\mathbf{D} \equiv \epsilon \mathbf{E}$.

Form this equation we infer that $\mathbf{k}$ and $\mathbf{E}$ are mutually perpendicular (why?). Thus, $\rho_e = 0$ (no charge separation) and $\nabla \cdot \mathbf{J_e} = 0$. However, $\mathbf{J_e}$ may be different from zero.

Dispersion relation for EM waves in cold plasma

Now we can plug $\mathbf{J_e}$ in Ampere's law (with displacement current):

$$\mathbf{k} \times \mathbf{B} = -\frac{\omega}{c}\epsilon\mathbf{E}$$

If we take the cross product of both sides of the equation and use the relationship $\mathbf{k} \times \mathbf{k} \times \mathbf{B} = \mathbf{k}(\mathbf{k} \cdot \mathbf{B}) - k^2\mathbf{B} = -k^2\mathbf{B}$ (see HW#5) we have

$$-k^2\mathbf{B} = -\epsilon\frac{\omega}{c}\mathbf{k} \times \mathbf{E} = -\epsilon\left(\frac{\omega}{c}\right)^2\mathbf{B}$$

which yields the desired *dispersion relation*:

$$\epsilon\omega^2 = k^2c^2 \quad \text{or} \quad \frac{\omega}{k} = \pm\frac{c}{\epsilon^{1/2}} = \pm\frac{c}{n_r}$$

where we have defined the index of refraction $n_r \equiv \epsilon^{1/2}$.

Physical interpretation of plasma frequency

Clearly if $\epsilon < 0$ then the wave number $\mathbf{k}$ is imaginary: the wave is exponentially damped. This is call “evanescent wave” since there is not dissipation or absorption of the EM wave.

Let’s first consider the case of weakly magnetized plasma: $\omega_L \ll \omega$.

In this case $\epsilon = 1 - (\omega_{pe}/\omega)^2$.

Waves with frequency $\omega < \omega_{pe}$ cannot propagate in the medium and will be reflected (ω_{pe} is also known as plasma cutoff frequency).

Earth’s atmosphere has $n_e \sim 10^6$ cm^{-3}, $\lambda_{pe} = 2\pi c/\omega_{pe} \sim 30$ meters.

What is the physical interpretation? Let’s consider the longitudinal displacements of electrons (why?) in response to propagation of EM wave.

Wave travels toward the ceiling. Electric field is in the x-direction:

+	+	-	+	-	+	-	+	-
+	-	+	-	+	-	+	-	-
+	+	-	+	-	+	-	+	-
+	-	+	-	+	-	+	-	-
+	0	0	0	0	0	0	0	-

This looks like a capacitor. Electric field: $E_x = 4\pi e n_e x$. Equation of motion of free electrons:

$$m_e \ddot{x} = -eE_x = -m_e \omega_{pe}^2 x$$

What is this?

In Maxwell's equation displacement current is $\partial E/\partial t \sim -i\omega E_x$. Conduction current $4\pi J_e \sim -4\pi e n_e \dot{x} = -\dot{E}_x = i\omega_{pe} E_x$. Thus, the sum is zero for $\omega = \omega_{pe}$. EM wave cannot propagate!

Application I. Pulsar dispersion measure

$v_f = \omega/k = c/\epsilon^{1/2}$ can be $> c$!
No problem (why?). The group velocity is

$$v_g = \frac{\partial \omega}{\partial k} = c/\epsilon^{1/2} - \left(\frac{ck\omega_{pe}^2}{\epsilon^{3/2}\omega^3} \right) \frac{\partial \omega}{\partial k} = c/\epsilon^{1/2} - v_g(1-\epsilon)/\epsilon \rightarrow v_g = c\epsilon^{1/2} < c$$

Let's consider a radio pulse from a pulsar at distance r. The travel time is:

$$t_\omega = \int_0^r \frac{ds}{v_g} = \int_0^r \frac{ds}{c} \epsilon^{-1/2} \sim \frac{r}{c} + \frac{2\pi e^2}{m_e \omega^2} \mathrm{DM}$$

where we have assumed $\epsilon^{-1/2} \sim 1 + (\omega_{pe}/\omega)^2/2 \sim 1$ and we have defined the *dispersion measure* of the medium:

$$\mathrm{DM} \equiv \int_0^r n_e ds.$$

This is the column density of free electrons. Thus $\langle n_e \rangle \equiv \mathrm{DM}/r$.

Application II: Faraday rotation

Let's now consider the case in which B_0 is not negligible. In this case the dispersion relation depends on the verse of polarization of the radiation. This means that left- and right- circular polarized waves travel in the medium at different speed. For $\omega \gg \omega_{pe}$ we have:

$$\epsilon_{\pm}^{1/2} = 1 - \frac{1}{2}\frac{\omega_{pe}^2}{\omega^2}\left(1 \pm \frac{\omega_L}{\omega}\right) = 1 - \frac{1}{2}\frac{\omega_{pe}^2}{\omega^2} \mp \frac{1}{2}\frac{\omega_L \omega_{pe}^2}{\omega^3}$$

Thus the dispersion relation can be written as the sum of two terms,

$$k_{\pm} = \epsilon_{\pm}^{1/2}\frac{\omega}{c} = k_0 \mp \Delta k$$

where

$$k_0 \equiv \frac{\omega}{c}\left(1 - \frac{\omega_{pe}^2}{2\omega^2}\right), \quad \text{and} \quad \Delta k \equiv \frac{\omega_{pe}^2 \omega_L}{2\omega^2 c}$$

where the expression for k_0 is the previously found dispersion relation

(in absence of magnetic field).

NOTE: The magnetic field that is relevant in the problem (in ω_L) is the component of $\mathbf{B}$ parallel to the direction of propagation of the wave.

Let's consider a wave that is emitted at the source with net linear polarization along the x-direction:

$$\mathbf{E} = \hat{\mathbf{x}}\mathcal{E}_0 e^{-i\omega t} \equiv \frac{1}{2}[(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})]\mathcal{E}_0 e^{-i\omega t}$$

After propagating a distance r (along the z-axis) trough a magnetized plasma toward the observed, the electric field behaves as

$$\mathbf{E} = \hat{\mathbf{x}}\mathcal{E}_0 e^{-i\omega t} \equiv \frac{1}{2}[(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{\int_0^r (k_0+\Delta k)dz} + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{\int_0^r (k_0-\Delta k)dz}]\mathcal{E}_0 e^{-i\omega t}$$

Let's define $\varphi \equiv \int_0^r k_0 dz$, and $\psi \equiv \int_0^r \Delta k dz = \frac{2\pi e^3}{m_e^2 c^2 \omega^2}\mathrm{RM}$ where the "rotation measure" is defined

$$\mathrm{RM} \equiv \int_0^r n_e B_{||} ds$$

Factoring the common phase $e^{i\varphi}$ and expanding $e^{\pm\psi}$ we obtain

$$\mathbf{E} = (\hat{\mathbf{x}} \cos\psi + \hat{\mathbf{y}} \sin\psi)\mathcal{E}_0 e^{\varphi - i\omega t}$$

Thus, the orientation of the linearly polarized radiation rotates by an angle ψ. This angle depends on the frequency of radiation and the RM. We can use measurements of linearly polarized radiation at different frequencies to measure RM. If DM is also measure we can derive the mean strength of the magnetic field along the line of sight:

$$\langle B_{||} \rangle = \frac{\mathrm{RM}}{\mathrm{DM}} \sim 3\mu G \;\; \text{in the ISM}$$

Application to the ISM in our Galaxy:

$$U_B = \frac{3B_{||}^2}{8\pi} \sim 1 \times 10^{-12} erg/cm^3 \sim 1\; eV/cm^3$$

We have roughly equipartition of energy between magnetic field, CR, turbulent and thermal pressure: $U_B \sim U_{CR} \sim U_{th} \sim U_{turb}$.

Cherenkov Radiation

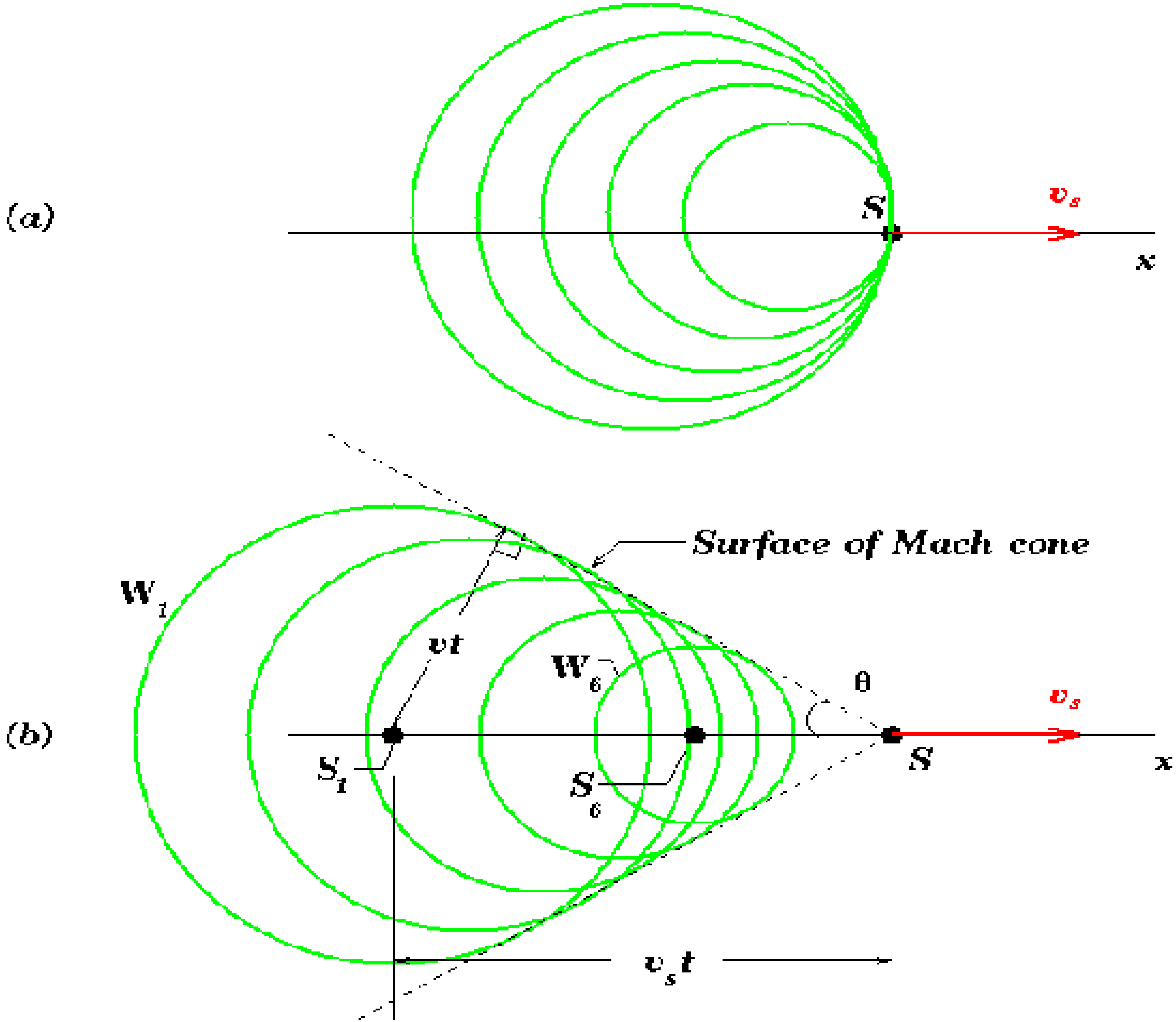
Radiation from relativistic charges moving in a plasma with $n_r > 1$. In this case the velocity of the charges can exceed the phase velocity $v_f = c/n_r$.

The Lienard-Wiechert potentials still give the $\mathbf{E}$ and $\mathbf{B}$ field for a moving charge with replacement of $c \to c/n_r = v_f$, $\mathbf{E} \to n_r \mathbf{E}$, $e \to e/n_r$.

Thus, the beaming term $\mathcal{K} = 1 - (v/v_f)\cos\theta$ may vanish. In consequence a uniformly moving particle can now radiate.

Also, if the velocity of the particle is $v > v_f$ the potential at a point may be determined by two retarded positions of the particle (not just one).

The space is divided in 2 regions: inside the Cherenkov cone where each point feels the potential from two retarded positions of the particle, and outside. Cherenkov radiation is confined within the cone.


(a)
S
v_s
x
(b)
Surface of Mach cone
W_1
vt
W_6
θ
v_s
S_1
S_6
S
x
$v_s t$

VERITAS Cherenkov telescope

Razin Effect

When $n_r < 1$ in a cold plasma, Cherenkov radiation cannot occur. However, there is an effect important for synchrotron radiation (need thermal plasma together with relativistic CRs).
The beaming effect, important for synchrotron radiation may vanish if the opening angle is never small:

$$\theta \sim \frac{1}{\gamma} \sim \sqrt{1 - n_r^2 \beta^2}$$

There are two regimes. At low frequencies n_r is small and we have

$$\theta \sim \sqrt{1 - n_r^2} \sim \frac{\omega_{pe}}{\omega}.$$

At high frequencies $n_r \sim 1$ and we get $\theta = 1/\gamma$.
Thus, if $\omega \ll \gamma\omega_{pe}$ the medium may suppress the beaming effect that produces synchrotron radiation. This is known as the *Razin effect*.