Queen Mary
University of London

M.Sc./M.Sci. EXAMINATION BY COURSE UNITS

ASTM002 / MAS430 The Galaxy

Friday, 30th May, 2008
10:00 a.m. – 1:00 p.m.

Time Allowed: 3 hours

You may attempt as many questions as you wish and all questions carry equal marks. Except for the award of a bare pass, only the best 5 questions answered will be counted.

Calculators ARE permitted in this examination. The unauthorised use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

YOU ARE NOT PERMITTED TO START READING THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR

Useful information

In this paper π and e represent the conventional mathematical constants.
G represents the gravitational constant with $G = 6.67 \times 10^{-11}\ \mathrm{m^3\,kg^{-1}s^{-2}}$.
c is the velocity of light, with $c = 3.00 \times 10^8\ \mathrm{m\,s^{-1}}$.
1 pc $= 3.09 \times 10^{16}$ m.
The distance of the Sun from the Galactic Centre is $R_0 = 8.0\,\mathrm{kpc}$.

Poisson's equation states that $\nabla^2\Phi = 4\pi G\rho$ at any point in a gravitational field, where Φ is the gravitational potential, G is the constant of gravitation, and ρ is the density at that point.

The Laplacian of a scalar function Φ in a spherical coordinate system (r, θ, ϕ) is

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} .$$

The Laplacian of a scalar function Φ in a cylindrical coordinate system (R, ϕ, z) is

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2} .$$

The Jeans equations in a steady-state (i.e. time-independent), spherically-symmetric galaxy give the following result

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(n\,\langle v_r^2\rangle\right) + \frac{n}{r}\left[\,2\langle v_r^2\rangle - \langle v_\theta^2\rangle - \langle v_\phi^2\rangle\,\right] = -\,n\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} ,$$

in spherical coordinates, where n is the number density of stars at a distance r from the centre, v_r, v_θ and v_ϕ are the components of the velocity in the r, θ and ϕ directions, and $\Phi(r)$ is the gravitational potential.

The virial theorem states that the time-averaged total kinetic energy $\langle T\rangle$ of a system of particles in equilibrium is related to the time-averaged total internal potential energy $\langle U\rangle$ by $2\langle T\rangle + \langle U\rangle = 0$.

In the absence of cosmological effects, the apparent magnitude m of an astronomical object in a photometric band is related to its absolute magnitude M in that band and its distance D from the observer by

$$m - M = 5\log_{10}(D/\mathrm{pc}) - 5 + A ,$$

where A is the extinction in the band expressed in magnitudes.

The fundamental plane for elliptical galaxies is approximately

$$R_S\, I_0^{0.8}\, \sigma_0^{-1.3} = \text{constant} ,$$

where R_S is the scale size of a galaxy, I_0 is its central surface brightness, and σ_0 is the line-of-sight velocity dispersion at its centre.

Oort's constants within the Galaxy are defined as

$$A \;\equiv\; \frac{1}{2}\left(\frac{\langle v_\phi\rangle}{R} - \frac{\partial\langle v_\phi\rangle}{\partial R}\right) \qquad \text{and} \qquad B \;\equiv\; -\frac{1}{2}\left(\frac{\langle v_\phi\rangle}{R} + \frac{\partial\langle v_\phi\rangle}{\partial R}\right) \;,$$

where $\langle v_\phi\rangle$ is the mean tangential velocity in the Galactic disc, and R is the distance from the Galactic Centre, for $R = R_0$.

A standard integral: $$\int \frac{x^2}{(x^2+a^2)}\,\mathrm{d}x \;=\; x \;-\; a\tan^{-1}\!\big(x/a\big) \;+\; \text{constant}\,.$$

[End of the useful information]

1. (a) Briefly describe the observed properties of elliptical galaxies. (You should include reference to their general structure, their stars, gas content, spectra and colours.) [5 marks]

(b) Show that the fundamental plane for elliptical galaxies implies that the luminosity of a galaxy $L \propto \sigma^{2.6}/I_0^{0.6}$, where I_0 is its central surface brightness and σ_0 is the line of sight velocity dispersion at its centre. [3]

(c) The spectrum of a distant galaxy shows a strong continuum with absorption lines, and some emission lines superimposed. What is the morphological type of the galaxy likely to be? Explain your reasoning. [2]

(d) A large stationary gas cloud collapses under its own gravitation. 30 % of the initial potential energy is converted into heat, raising the temperature of the gas. Is this collapse dissipative or dissipationless? [2]

(e) What value does the sum, $A+B$, of the Oort constants have if the rotation curve of the Galaxy is flat near the Sun? [2]
Express the angular velocity of the Galactic disc at the Sun's distance from the centre in terms of A and B. [1]

(f) Explain the term *asymmetric drift.* [2]
How does the asymmetric drift vary with age for stars in the Galactic disc? How does the velocity dispersion of stars depend on age? What effect might giant molecular clouds in the disc of the Galaxy have had on the velocity dispersions of old stars? [3]

[Total 20 marks for question]

2. (a) Show that in a weak encounter between two stars of mass m the change in the velocity v of one star in the reference frame of the other is given by

$$\delta v \;=\; \frac{2Gm}{bv} \;,$$

[This question continues overleaf . . .]

where G is the constant of gravitation and b is the impact parameter. You may assume that $\int_{-\infty}^{\infty} (k_1 + k_2 s^2)^{-3/2}\, \mathrm{d}s = 2/(k_1\sqrt{k_2})$. [8 marks]

A star moves through a spherical distribution of N stars of overall radius R, with the stars distributed uniformly in space. If the change in the square of the velocity is $\delta v^2 = (\delta v)^2$ in a weak encounter, show that the changes in v^2 caused by all encounters with impact parameters in the range b to $b + \mathrm{d}b$ in a time t is

$$\Delta v^2 \;=\; \left(\frac{2Gm}{bv}\right)^2 \left(\frac{3bvtN\mathrm{d}b}{2R^3}\right) \;. \qquad [4]$$

Hence show that the total change in the square of the velocity in a time t caused by weak encounters with all impact parameters is

$$\Delta v^2(t) \;=\; 6\left(\frac{Gm}{v}\right)^2 \frac{v\,t\,N}{R^3} \ln\left(\frac{b_{max}}{b_{min}}\right) \;,$$

where b_{max} and b_{min} are the largest and smallest values of the impact parameter. [4]

Thereby derive the expression for the relaxation time

$$T_{relax} \;=\; \frac{1}{6N\ln\left(\frac{b_{max}}{b_{min}}\right)} \frac{(Rv)^3}{(Gm)^2} \;. \qquad [4]$$

[Total 20 marks for question]

3. (a) Explain briefly what is the distribution function f that used in studying the dynamics of galaxies? [2 marks]

(b) A star moves in an elongated orbit about the Galaxy. As it moves inwards its distance from the Galactic Centre decreases and the star density around it increases. How does f change? How do the velocity dispersions of the stars around it change and why? [3]

(c) The distribution function f in a spherically-symmetric galaxy is related to the mass density $\rho(r)$ at a radial distance r from the centre by

$$\rho(r) \;=\; 4\pi\sqrt{2}\,\overline{m} \int_{\Phi(r)}^{0} \sqrt{E_m - \Phi(r)}\; f(E_m)\, \mathrm{d}E_m \;,$$

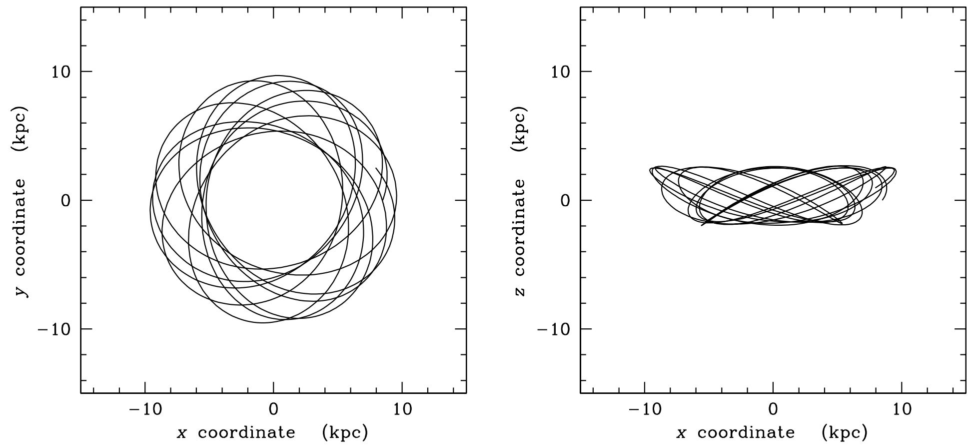
where E_m is the energy per unit mass of a star, $\Phi(r)$ is the gravitational potential at a radius r, and $\overline{m}$ is the mean mass per star. A spherical galaxy is modelled using a potential $\Phi(r)$ and density $\rho(r)$ given by

$$\Phi(r) \;=\; -\frac{GM_{tot}}{\sqrt{r^2 + a^2}} \quad \text{and} \quad \rho(r) \;=\; \frac{3M_{tot}}{4\pi} \frac{a^2}{(r^2 + a^2)^{5/2}} \;,$$

where M_{tot} is the total mass of the galaxy and a is a constant.
Show that a functional form $f(E_m) = b\,(-E_m)^{7/2}$ is a solution for the distribution function for this model where b is a constant. The substitution $E_m = \Phi\cos^2\theta$ and the standard result $\int_0^{\pi/2} \sin^2\theta\, \cos^8\theta\, \mathrm{d}\theta = 7\pi/512$ may prove useful. [9]

[This question continues overleaf . . .]

(d) The diagrams below show the orbit of a star in a gravitational potential, shown projected on to the $x - y$ and the $x - z$ planes.



What do you conclude about the potential: is it (i) spherical, (ii) flattened (oblate), or (iii) triaxial? Justify your answer on the basis of the character of the orbit. [4]

(e) Explain very briefly the concept of violent relaxation. [2]

[Total 20 marks for question]

4. (a) What advantage do the Jeans equations have over the collisionless Boltzmann equation in describing the dynamics and densities of stars in observed galaxies? [2 marks]

(b) The collisionless Boltzmann equation gives

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} \left(\frac{\mathrm{d}x_i}{\mathrm{d}t} \frac{\partial f}{\partial x_i} + \frac{\mathrm{d}v_i}{\mathrm{d}t} \frac{\partial f}{\partial v_i} \right) = 0 ,$$

where f is the distribution function, t is time, and x_i and v_i (for $i = 1, 2, 3$) are the components of the position vector $\mathbf{x}$ and velocity vector $\mathbf{v}$ respectively. Derive from this the second Jeans equation,

$$\frac{\partial (n\langle v_i \rangle)}{\partial t} + \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \big(n \langle v_i v_j \rangle \big) = - \frac{\partial \Phi}{\partial x_i} n \ ,$$

where $i = 1, 2$ or 3, $\Phi(\mathbf{x}, t)$ is the gravitational potential, and $n(\mathbf{x}, t)$ is the number density of stars at a point in space. For this you may assume that

$$n\langle v_i \rangle = \int v_i \, f \, \mathrm{d}^3\mathbf{v} \, , \qquad n\langle v_i v_j \rangle = \int v_i \, v_j \, f \, \mathrm{d}^3\mathbf{v} \, ,$$

$$\text{and} \qquad \int v_i \frac{\partial \Phi}{\partial x_j} \frac{\partial f}{\partial v_j} \, \mathrm{d}^3\mathbf{v} = - \frac{\partial \Phi}{\partial x_j} \delta_{ij} \, n \ ,$$

[This question continues overleaf . . .]

where δ_{ij} is the Kronecker delta function. Here $\langle p \rangle$ denotes the mean value of some quantity p. [8]

(c) A spherically-symmetric galaxy is dark-matter dominated and has a gravitational potential

$$\Phi(r) \;=\; -\frac{GM_{tot}}{r+a}$$

at a radial distance r from its centre, where a is a positive constant and M_{tot} is the total mass.

A population of stars is distributed within this potential. The stars contribute negligibly to the total density. The system of stars has an isotropic velocity distribution with a velocity dispersion σ that is constant across the galaxy, and has zero net rotation. Assuming that the potential is constant over time, derive an expression for the number density n of stars as a function of radius r and in terms of the number density n_0 of stars at the centre. [6]

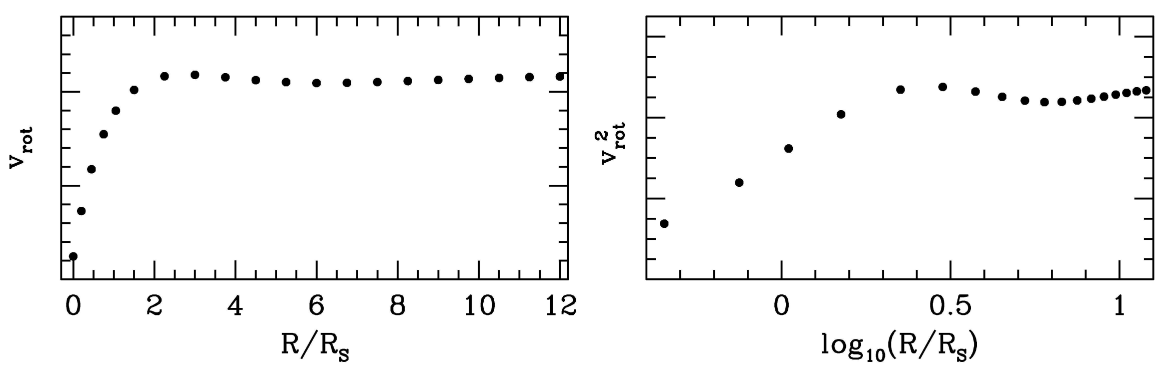
(d) A spherically-symmetric galaxy consists only of stars (it has no dark matter). It has a density distribution for which the internal potential energy is

$$U \;=\; -\frac{GM_{tot}^2}{6a} \;.$$

If the typical velocity of stars in this galaxy is v, calculate the total mass from the virial theorem, in terms of a and v, on the assumption that the system is pressure supported and virialised, and v is constant throughout. [4]

[Total 20 marks for question]

5. (a) The graphs below plot the rotation curve of a spiral galaxy. The first graph shows the circular velocity v_{rot} against R/R_S, where R is the radial distance from the centre and R_S is the exponential scale length of the galaxy's disc observed in visible light. The second plots v_{rot}^2 against $\log_{10}(R/R_S)$.



Identify the velocity maxima in the rotation curve. Which components of the galaxy show a dynamical signature: a bulge, a disc and/or a dark matter halo? Explain your reasoning. [6 marks]

In which part of the electromagnetic spectrum were the observations made? Explain your reasoning. [2]

[This question continues overleaf . . .]

(b) The dark matter within our Galaxy is sometimes modelled using a spherically-symmetric density distribution

$$\rho(r) \;=\; \frac{\rho_0}{1 + r^2/a^2}\,,$$

where $\rho(r)$ is the mass density at a distance r from the centre, and ρ_0 and a are positive constants. Show that the mass interior to a radius r is

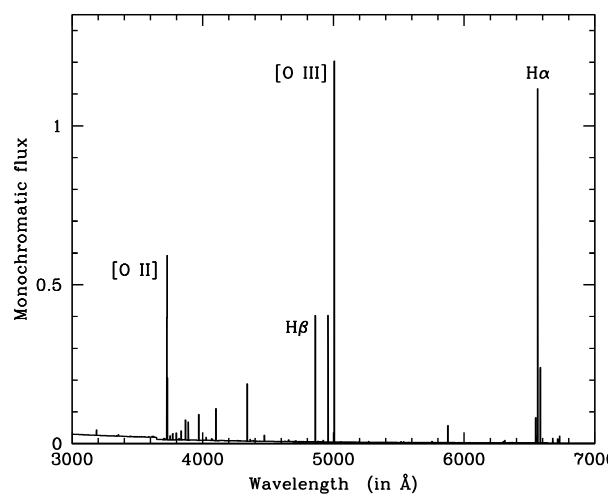
$$M(r) \;=\; 4\pi\rho_0\, a^2 \left(r \;-\; a\tan^{-1}(r/a)\right). \qquad [6]$$

Derive an expression for the circular velocity v_{circ} for this mass distribution. What is the dependence of v_{circ} on radial distance r at large distances ($r \gg a$)? How does this compare with the observed rotation curve of the Galaxy? [4]

What is the total mass out to infinity implied by this density profile? How must the real density profile of the Galaxy behave at large radii compared with this model profile? [2]

[Total 20 marks for question]

6. (a) The figure (right) shows the optical spectrum of the Orion Nebula. Briefly explain the different emission lines and the physical mechanisms responsible for producing them. [4 marks]



(b) What physical mechanism is responsible for maser emission of radiation from the interstellar medium in the immediate vicinity of some radiation sources? [2]

(c) Is gas collisional or collisionless in an astronomical context? [2]

(d) A star lying in the Galactic plane is observed to have a visual magnitude of $V = 17.15$ and a blue magnitude of $B = 18.45$. Spectroscopy shows the star to be of a type that has an intrinsic colour $(B-V)_0 = 0.80$ mag and a V-band absolute magnitude $M_V = +5.60$. What is the colour excess of the star? Estimate the extinction in the V band towards the star. Estimate the distance of the star from the Sun.

If the star has a Galactic longitude of $l = 180°$, estimate its distance from the Galactic Centre. [5]

[This question continues overleaf . . .]

(e) The mean density in the form of stars in the disc of the Galaxy is observed to vary with the distance z from the Galactic plane as $\rho_s(z) = \rho_{so} e^{-|z|/h_s}$ close to the Sun, where ρ_{so} is the density of stars in the plane, and h_s is a scale height. The density of the interstellar gas ρ_g is also found to vary roughly exponentially with height with $\rho_g(z) = \rho_{go} e^{-|z|/h_g}$, where ρ_{go} and h_g are constants. Observations show that $h_s = 250\,\mathrm{pc}$, $h_g = 150\,\mathrm{pc}$ and $\rho_{so} = 6\,\rho_{go}$.

What is the ratio of the surface density of stars, Σ_s, to that of gas, Σ_g, at the Sun's distance from the Galactic Centre? [4]

How do you expect the surface density of the dust, Σ_d, to compare with Σ_s? [1]

How realistic is this representation of the gas density as $\rho_g(z) = \rho_{go} e^{-|z|/h_g}$ in practice? [2]

[Total 20 marks for question]

7. (a) List any four assumptions behind the Simple Model of galactic chemical evolution. [4 marks]

(b) In a region of the Galaxy, the total mass of stars is $M_{\rm stars}$, the total mass of interstellar gas is $M_{\rm gas}$, and the mass of heavy elements in the interstellar medium is $M_{\rm metals}$, while the metallicity of the gas is Z. The changes in these quantities in a small time interval are $\delta M_{\rm stars}$, $\delta M_{\rm gas}$, $\delta M_{\rm metals}$ and δZ respectively.
For the Simple Model of galactic chemical enrichment derive the expression

$$\delta Z = \frac{\delta M_{\rm metals}}{M_{\rm gas}} - Z\,\frac{\delta M_{\rm gas}}{M_{\rm gas}} \,. \qquad [4]$$

If $\delta M_{\rm metals}$ and $\delta M_{\rm stars}$ are related by $\delta M_{\rm metals} = -\,Z\,\delta M_{\rm stars} + p\,\delta M_{\rm stars}$, where p is the yield of heavy elements, show that

$$\delta Z = -\,p\,\frac{\delta M_{gas}}{M_{gas}} \,. \qquad [3]$$

Hence show that the metallicity Z of the interstellar medium is related to the gas fraction μ by

$$Z = -\,p \ln \mu \,. \qquad [4]$$

How well does this prediction match observations of the gas in galaxies? [3]

(c) A Galactic star is observed to have [Fe/H] = −2.0. To which component of the Galaxy is it likely to belong? How old is it likely to be? [2]

[Total 20 marks for question]

[Next question overleaf]

8. (a) The angular size corresponding to the Einstein radius of a gravitational lens is

$$\theta_E \;=\; \sqrt{\frac{4GM_L}{c^2}\frac{D_{LS}}{D_L D_S}} \quad ,$$

where M_L is the mass of the lens, and D_S, D_L and D_{LS} are the distances from the observer to the light source, from the observer to the lensing object, and from the lens to the source respectively. Show that the optical depth through a distribution of microlenses of mass M_L along a path length to a source is given by

$$\tau \;=\; \frac{4\pi G}{c^2 D_S}\int_0^{D_S} D_L\, D_{LS}\, \rho(D_L)\, \mathrm{d}D_L \quad ,$$

where ρ is the mean mass density of lenses. [6 marks]

(b) A survey attempts to detect microlensing events from MACHOs by observing a field in the Galactic Bulge close to the Galactic Centre. Assume that the dark matter halo is made from compact objects (MACHOs) with approximately stellar mass and with a density distribution

$$\rho(r) \;=\; \frac{\rho_0\, a^2}{r^2 + a^2} \quad ,$$

where r is the radial distance from the Galactic Centre, ρ_0 is the central dark matter density and a is a constant. Show therefore that the optical depth of microlensing to the field is

$$\tau \;=\; \frac{2\pi G \rho_0 a^2}{c^2}\left(\ln\left(1 + \frac{R_0^2}{a^2}\right) \;+\; \frac{2a}{R_0}\tan^{-1}\frac{R_0}{a} \;-\; 2\right) \, ,$$

where R_0 is the distance of the Sun from the Galactic Centre. You may assume that the star field is not significantly affected by dust extinction for this calculation. You may find helpful the standard integral

$$\int \frac{x\,(b-x)}{(b-x)^2 + a^2}\,\mathrm{d}x \;=\; -x - a\tan^{-1}\left(\frac{b-x}{a}\right) - \frac{1}{2}b\ln\left(a^2 + (b-x)^2\right) \;+\; \text{constant}.$$

[8]

Estimate τ to within an order of magnitude if $R_0 = 8.0$ kpc, $a = 2.0$ kpc and $\rho_0 = 2.0 \times 10^{-20}\,\mathrm{kg\,m^{-3}}$. What does this imply for the number of stars that would have to be studied in the microlensing survey? [3]

(c) Draw a simple sketch illustrating how the brightness of a distant star varies with time before, during and after a microlensing event. [3]

[Total 20 marks for question]

[End of examination paper]